From Archimedes and Euclid to Hamilton and Poincaré

MATHEMATICAL CURIOSITIES

By Mark Levi

Symplectic maps of $\mathbb{R}^{2n}$ are basic objects of Hamiltonian mechanics, and the time t map of a Hamiltonian system's position-momentum pair is symplectic. Symplectic maps of $\mathbb{R}^2$ are the area-preserving ones.

I recently realized that the Archimedian law of the lever amounts to an area-preservation property of a simple map of $\mathbb{R}^2$, as described next. Afterwards, I will reference an analogy between the Archimedian lever on one hand and the Hamiltonian mechanics on the other.

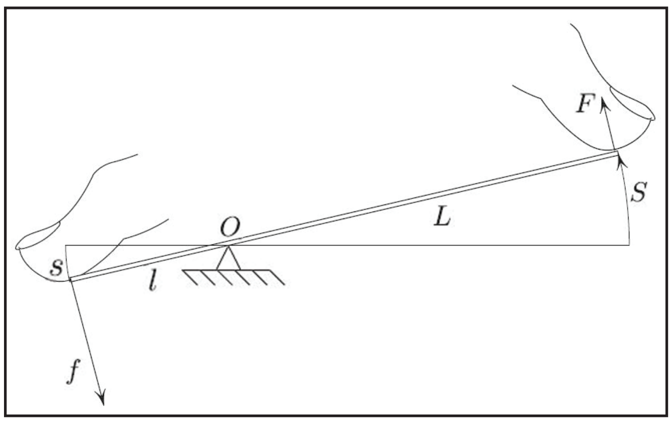

Figure 1. *The left finger pushes with force f; the right finger is being pushed with force F.*

Figure 1 shows a seesaw in equilibrium, pressed at both ends. Archimedes' law of the lever gives the condition for the equilibrium, $fl = FL$, i.e.,

$$F = \frac{l}{L} f.$$

Furthermore, $S = \frac{L}{l} s$, according to Euclid. To summarize, we have the "Euclid-Archimedes map," $(s, f) \mapsto (S, F)$, given by

$$\begin{cases} S = \lambda s \\ F = \dfrac{1}{\lambda} f, \end{cases} \qquad (1)$$

where $\lambda = L/l$. This map is clearly area-preserving, but for a reason deeper than the explicit form (1). Indeed, let us cyclically move the two fingers in Figure 1 so that the positions and the forces return to their original values. We end up doing zero work:

$$\oint f\, ds + \oint (-F)\, dS = 0. \qquad (2)$$

The minus sign is due to the fact that the right finger presses with force $-F$. During the cyclic motion, the point (s, f) describes a closed curve γ in the plane, while the point $(S, F) = \varphi(s, f)$ describes the image curve $\varphi(\gamma)$. Therefore, the zero-work condition (2) amounts to the equality of areas inside γ and $\varphi(\gamma)$. Incidentally, (2) is a compact way of saying that the lever is not a perpetual motion machine.

If the board can flex, as in Figure 2, then the map $(s, f) \mapsto (S, F)$ is no longer given by (1), but is still area-preserving; the above proof applies without change.

Seesaw and Hamiltonian Dynamics

Remarkably, the Hamiltonian flow is symplectic for the same reason that the "seesaw map" φ is area-preserving.[1] To make sense of the last sentence, I must specify the analogy between the seesaw in Figure 2 on one hand and a Hamiltonian system on the other. The following explanation outlines this analogy (a full discussion can be found in [1]). Consider a mechanical system with the Lagrangian L, depending on generalized position and velocity. Let us fix two points $(0, q)$ and (T, Q) in time-space and define the action $A(q, Q) = \int_0^T L(r(t), \dot{r}(t))\,dt$, with the integration occurring over the minimizer $r(t)$ of the integral subject to $r(0) = q$, $r(T) = Q$ (we assume this minimizer is unique and depends smoothly on q, Q). For any (admissible) T, the momenta at times $t = 0$ and $t = T$ are given by

$$P(T) = A_Q(q, Q), \quad p(0) = -A_q(q, Q). \qquad (3)$$

This can be taken as the definition of the momentum, or related (in a one-line calculation) to the more standard definition, as explained in page 261 of [1].

Returning now to the seesaw of Figure 2, let $U(s, S)$ be the potential energy; then

$$F = U_S(s, S), \quad f = -U_s(s, S). \qquad (4)$$

A comparison between (3) and (4) shows that the action and the momenta (A, p, P) are close analogs of the potential energy and the forces (U, f, F). The proof of the symplectic character of the time T map $(q, p) \mapsto (Q, P)$ for arbitrary T becomes a verbatim copy of the area preservation's proof of the "seesaw map" $(s, f) \mapsto (S, F)$.

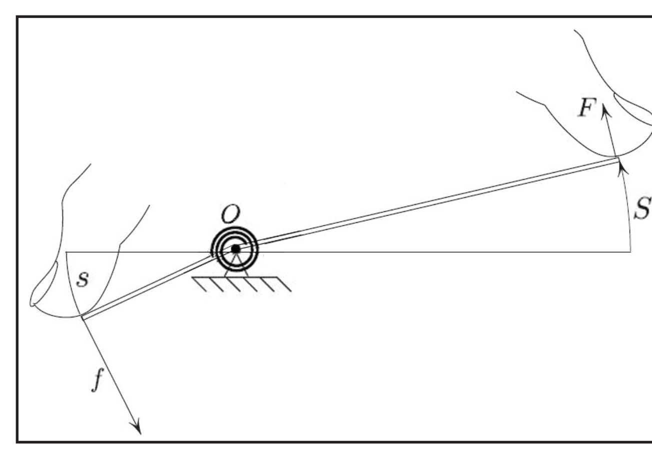

Figure 2. *The hinge at O has a spring trying to keep the board straight.*

A Paradox

If the spring in Figure 2 dissipates energy under deformations, then (2) becomes

$$\oint f\, ds + \oint (-F)\, dS = W > 0, \qquad (5)$$

where W is the heat dissipated in the spring x; (5) suggests that the area decreased by W. However, the map $\varphi := (s, f) \mapsto (S, F)$ depends only on the static properties of the spring and thus *must* be area-preserving; there is no difference between a dissipating and a non-dissipating spring in a static state. Resolution of this paradox is left as a puzzle for interested readers and may (or may not) be discussed in the next column.

All figures in this article are provided by the author.

Acknowledgments: The work from which these columns are drawn is partially supported by NSF grant DMS- 9704554.

References

[1] Levi, M. (2014). *Classical Mechanics with Calculus of Variations and Optimal Control: an Intuitive Introduction.* Student Mathematical Library, vol. 69. American Mathematical Society.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.

[1] or symplectic, if we allow more than one degree of freedom to move the endpoints.