On Littlewood's counterexample of unbounded motions in superquadratic potentials.

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1. Introduction.

Littlewood [10] constructed an example of an equation of the form $\ddot{x} + V'(x) = p(t)$ with $\frac{V'(x)}{x} \to \infty$ and yet possessing unbounded solutions. This note contains a considerable simplification of Littlewood's construction and gives rather precise information on the behavior of $\frac{V'(x)}{x} \to \infty$ under which the resonances of the type constructed by Littlewood are possible.

The above equation is a model Hamiltonian system for which the problem of stability (i.e. of the boundedness of solutions for all time) arises in its full difficulty. The Poincaré period map (also called the stroboscopic map) which maps the plane of initial conditions $(x, \dot{x})$ into itself* is area–preserving due to the conservative character of the equation. If the potential $V(x)$ happens to be superquadratic, i.e., if it grows faster than x^2 as $|x| \to \infty$, then one might expect that map possesses a twist at infinity, i.e. that the the points that are further from the origin undergo more revoluitons during one period of the forcing. In physical rather than the geometrical terms, the twist corresponds to the dependence of the frequency on the amplitude. Intuitively speaking, the twist may be expected to destroy the resonances that could cause the accumulation of energy and thus the unboundedness. By contrast, in the case of *quadratic* potential, the equation is linear: $\ddot{x} + \omega^2 x = p(t)$, and it is well known that the parametric resonance can occur, with the solutions growing without bound. In some examples of superquadratic potentials it has in fact been shown that all solutions stay bounded for all time [3], [4], [12] (see also [11], [17] for the case of potentials periodic in x). The proofs of boundedness all use Moser's twist theorem [7], [14], [18]. The Poincaré–Birkhoff fixed point theorem [2], [5], [9] has also been used to show the existence of periodic solutions [6], [8]. The Aubry–Mather theory and other recent results on monotone twist maps [1], [9], [13], [15], [16] shed new light on the qualitative behavior of the problem; we do not discuss here the implications of these results for our problem.

2. Results.

Returning to the discussion of the construction of an unbounded solution, we will start with the equation

$$\ddot{x} + 4x^3 = p(t) \tag{1}$$

where $p(t) = (-1)^{[t]+1}$, with $[t]$ denoting the integer part of t.

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* If the solutions do not blow up in finite time, as happens, for instance, to all solutions of $\ddot{x} - x^2 = 1$ at both ends of the time interval.

Theorem. *There exists a C^∞ modification $V(x)$ of the potential $V_0(x) = x^4$ satisfying for some $C > c > 0$ the estimates*

$$cx^4 \leq V(x) \leq Cx^4, \quad cx^2 \leq |V'(x)| \leq C|x|^3,$$

such that the equation

$$\ddot{x} + V'(x) = p(t) \tag{2}$$

has an unbounded solution, whose rate of growth, moreover, is given by

$$ct^{\frac{1}{3}} < (\dot{x}^2 + x^4)^{\frac{1}{4}} < Ct^{\frac{1}{3}},$$

where c, C are some constants which can be estimated more explicitly.

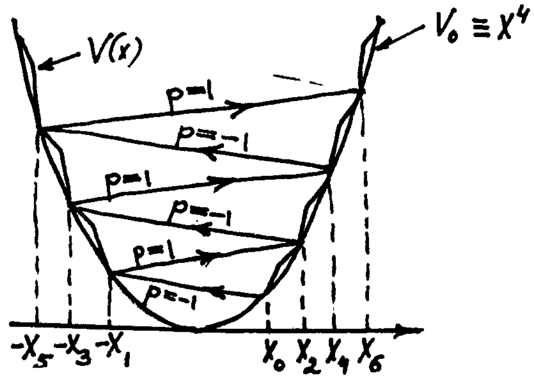

Figure 1. Resonance conditions and the modification of the potential.
Each arrow represents an odd number of swings which takes exacly one half–period.
If the graph of $V_0 \equiv x^4$ is replaced by $U_0 = V_0(x) \pm x$,
then the arrows marked $p = \mp 1$ become horizontal.

Remark 1. A twist criterion.

We will derive here a simple sufficient condition on the potential for the Poincare map of an *autonomous* system to possess twist, and show that this condition is violated by the potential constructed here. Let $z(t) = (x(t), \dot{x}(t))$ be a solution of $\ddot{x} + V'(x) = 0$ in the phase plane, and let $\zeta = (\xi(t), \eta(t))$ be a solution of the system linearized around $z(t)$. Assume that each solution vector $z(t)$ rotates clockwise in the phase plane. The criterion is based on the following geometrical observation, figure 2.

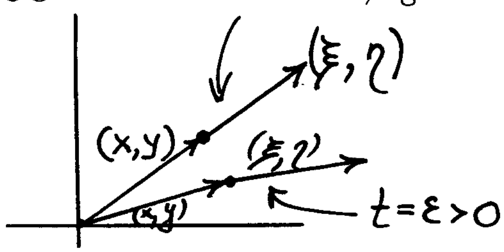

Figure 2. A sufficient condition for monotone twist:
any solution vector turns slower than the
collinear to it linearized solution vector.

Take any solution $z(t)$ and let $\zeta(0)$ be parallel to $z(0)$. If ζ turns clockwise faster than $z(t)$, then the period $T(E)$ of the oscillations is a decreasing function of the energy (or the amplitude, or the area enclosed by the curve). In other words, then the Poincare map posesses a monotone twist (with radial lines as the reference foliation). Expressing the above idea analytically, we rewrite the condition on the angular velocity in the form

$$\frac{d}{dt}\left(\frac{y}{x}\right) > \frac{d}{dt}\left(\frac{\eta}{\xi}\right), \quad \textbf{whenever} \quad \frac{y}{x} = \frac{\eta}{\xi}.$$

Using the governing equations $\dot{x} = y, \dot{y} = -V'(x), \quad \dot{\xi} = \eta, \dot{\eta} = -V''(x)\xi$, the above criterion reduces to

$$\frac{V'}{x} < V''.$$

This twist condition is violated by the modifications of the potential below, as one can see easily from the fact that V'' changes sign as $x \to \infty$ while V' does not.

Physical interpretation of the above twist criterion. We will give a the physical meaning to both sides of the last inequality. To that end, we think of the equation $\ddot{x} = -V'(x)$ as governing the oscillations of a particle moving along the x–axis subject to the restoring force $-V'$ of a nonlilnear spring. Geometrical property of twist translates into the physical property of the frequency of oscillations growing monotonically with the amplitude. Since there is zero twist for linear springs ($V' = const \cdot x$), we expect there to be a twist for springs for which V' is superlinear; the last word is in fact made precise by the last inequality, which we finally proceed to interpret physically. Fixing x_0, we interpret $\frac{V'(x_0)}{x_0} \equiv k(x_0)$ as the "cumulative" Hooke's coefficient of the spring at x_0, i.e. as the Hooke's constant of an imagined linear spring whose force is is $k(x_0)x$ when the elongation is x. (In particular, the imagined spring has the same tension $V'(x_0)$ as the given one at $x = x_0$.) On the other hand, $V''(x_0)$ is the "local" Hooke's constant which measures the increment of the tesion force $V'(x)$ per unit change of x at $x = x_0$. We can now restate the above twist condition as saying:

"If the global Hooke's coefficient $\frac{V'(x_0)}{x_0}$ is greater than the local Hooke's coefficient $V''(x_0)$ for all x_0, then the period of oscillations is a monotonically decreasing function of the energy".

3. Proof of the theorem.

In the first subsection we outline the idea of the construction; the details are carried out in later subsections.

3.1. An outline of the construction.

Referring to Figure 1, we start with $x_0 > 0$ chosen so that the solution $x(t)$ with an initial condition $x(0) = x_0, \quad \dot{x}(0) = 0$ swings left and stops at $x < 0$ at $t = 1$, i.e. with $x(1) \equiv -x_1 < 0, \ \dot{x}(1) = 0$. We will modify $V_0(x) \equiv x^4$ in such a way that the solution $x(t)$ of the equation (2) is in resonance with $p(t)$:

$$x(2n) \equiv x_{2n} > 0, \ \dot{x}(2n) = 0, \qquad (3a)$$

$$x(2n+1) \equiv -x_{2n+1} < 0, \ \dot{x}(2n+1) = 0, \tag{3b}$$

for all integer $n \geq 0$. The modification of x^4 will preserve the values of V_0 at the points $(-1)^k x_k$ where the solution turns around, i.e. $V((-1)^k x_k) = x_k^4$ for all integer $k \geq 0$. Once the potential $V_0(x) = x^4$ has been modified to achieve the resonance conditions (3), the unboundedness of the solution $x(t)$ follows at once. Indeed, during the first half-periods, when $2n \leq t \leq 2n+1$, we have $p = -1$ and thus $\frac{\dot{x}^2}{2} + V(x) + x = \text{const.}$, while for the second half-periods $2n+1 \leq t \leq 2n+2$ we have $\frac{\dot{x}^2}{2} + V(x) - x = \text{const.}$. Using these conservation relations for the endpoints $t = 2n$ and $t = 2n+1$, we obtain:

$$V(x_{2n}) + x_{2n} = V(-x_{2n+1}) + (-x_{2n+1}),$$

$$V(-x_{2n+1}) - (-x_{2n+1}) = V(x_{2n+2}) - (x_{2n+2}),$$

both of which yield the same relation

$$x_{k+1}{}^4 - x_k{}^4 = x_k + x_{k+1}, \tag{4}$$

where $V((-1)^k x_k) = x_k{}^4$ was used. This difference equation says simply that the net gain in potential energy V during each half-period equals the product of the external force $p(t) = (-1)^{k+1}$ acting during k-th half-period and the net distance $x(k+1) - x(k) = (-1)^{k+1} x_{k+1} - (-1)^k x_k = (-1)^{k+1}(x_{k+1} + x_k)$ traveled during that half-period, i.e. $x_{k+1} + x_k$. This energy gain is the key idea of Littlewood's example. It should be pointed out that the difference equation (4) defines the sequence $x_k > 0$ uniquely for any $x_0 > 0$, so that from now on the x_k's are the fixed quantities. The following lemma implies in particular that eq.(4) and thus the resonance conditions (3) imply the unboundedness of $x(t)$.

Lemma 1. *Any positive sequence $\{x_k\}$ with $x_0 > 0$ defined by the recurrence relation (4) satisfies the following estimates, for some $C > c > 0$, with $c = c(x_0)$, $C = C(x_0)$:*

$$ck^{\frac{1}{3}} < x_k < Ck^{\frac{1}{3}}, \tag{5a}$$

$$x_{k+1} - x_k < Ck^{-\frac{2}{3}}, \tag{5b}$$

$$cn^{\frac{4}{3}} < \sum_0^n x_k < Cn^{\frac{4}{3}}, \tag{5c}$$

$$cx_k < x_{k+2}^4 - x_k^4 < Cx_k, \tag{5d}$$

$$1 < \frac{x_{k+1}}{x_k} < C, \tag{5e}$$

for all $k \geq 1$.

Proof of the lemma is given in section 3.5 below. At this stage we use only the implication that $x_k \to \infty$.

Remark 2. Avoiding virtually all computation, one can easily produce a modification of the potential x^4 satisfying the resonance conditions (4); this argument, however, does not resolve the main difficulty which is to estimate the minimal necesary size of the modification, and in particular to see if one can keep the property $\frac{V'}{x} \to \infty$. To describe the argument, we start with with eq. (1) and choose the initial position x_0 so that the solution makes exactly one swing during the first half of the period, arriving at $-x_1$. If the potential is left unmodified, the solution will make more than one full swing during the second half period: it will reach the point x_2 defined above at least once. The potential can be modified in the interval $[x_0, x_2]$ in such a way that the solution is slowed down so that the point x_2 is reached exactly at the end of the period. A simple argument based on the superquadratic character of x^4 shows that during the next half–period the solution will again make at least one trip from x_2 to $-x_3$; by modifying V in the interval $[-x_1, -x_3]$ (by making $V'(x_3)$ small enough) we slow the solution down so that $x(3) = -x_3$, etc., thus achieving the resonance conditions. This argument is, however, too crude as it gives no estimate on the how small a change of V would suffice to slow the solution. It may be pointed out also that speeding the solution rather than slowing it cannot produce the resonance condition.

3.2. Modification of the potential.

The potential $V_0 = x^4$ will be modified inductively on the intervals I_n given by $I_{2k} = [x_{2k}, x_{2k+2}]$ and $I_{2k+1} = [-x_{2k+3}, -x_{2k+1}]$, $k \geq 0$, by replacing x^4 on I_n by a piecewise linear function as shown in figure 1 and as described below. We concentrate on the case of an even interval, I_{2n}; odd intervals are treated in the same way. Assume that the modification of the potential has been carried out in all previous intervals I_k, $k \leq 2n$ in such a way as to achieve the resonance conditions (3) and satisfying the growth estimates stated in the theorem on the interval $[-x_{2n-1}, x_{2n}]$. We will change the potential x^4 into a piecewise linear continuous function on the interval I_{2n}, with the slopes σ_1 and σ on the left and the right halves of I_{2n} respectively. The slope σ will serve as the parameter with which we will achieve the resonance condition. To make the modification of the potential more explicit, we denote $\xi = x - x_{2n+2}$, $a = x_{2n+2} - x_{2n}$, $b = x_{2n+2}^4 - x_{2n}^4$, (subscripts for a and b are omitted), and define $V_\sigma(x) = x_{2n+2}^4 + W(\xi)$, where

$$W(\xi) = \begin{cases} \sigma\xi & \text{for } -\frac{a}{2} \leq \xi \leq 0 \\ -b + \sigma_1(\xi + a) & \text{for } -a \leq \xi \leq -\frac{a}{2} \end{cases};$$

the slope σ is the parameter with which we will control the arrival time at the end of the period, and $\sigma_1 = \frac{2b}{a} - \sigma$. This modification $V_\sigma(x)$ of the potential on I_{2n} is continuous; it can be smoothed to a C^∞ function in small neighborhoods of the corners while preserving the statement of Lemma 2 below. This smoothing is described in the proof of that lemma.

3.3. Proof of the resonance condition.

According to our inductive assumption, the resonance equations (3) hold for all x_k with $k \leq 2n+1$ and $V'(x)/x > cx$ in the interval $-x_{2n-1} < x < x_{2n}$, with some $c > 0$ independent of n. Our aim is to show that for some $\sigma = \sigma^*$ the next resonance condition at $t = 2n+2$ is satisfied. To that end, consider the vector $z_\sigma = (x_\sigma(2n+2), \dot{x}_\sigma(2n+2))$ at which the solution $(x_\sigma(t), \dot{x}_\sigma(t))$ in question finds itself at $t = 2n+2$ (after making many (when n is large) oscillations during the preceding time interval of length one). We have to chose $\sigma = \sigma^*$ so that z_{σ^*} lies on the positive x–axis. To prove that such a choice exists, we will show that z_σ makes one full revolution as σ decreases from $\sigma = \sigma_0 \equiv \frac{b}{a}$; denoting by $\tau(\sigma)$ the time of a one–way swing (which is one half of the period of oscillation of the solution of $\ddot{x} + V_\sigma(x) = 1$ starting at $(x, \dot{x}) = (-x_{2n+1}, 0)$), we observe that if for some $\sigma < \sigma_0$ the numbers of one–way swings differs by at least two, or more precisely, if

$$\frac{1}{\tau(\sigma_0)} - \frac{1}{\tau(\sigma)} \geq 2, \tag{6}$$

i.e.

$$\tau(\sigma) - \tau(\sigma_0) \geq 2\tau(\sigma)\tau(\sigma_0), \tag{7}$$

then for some σ^* in $\sigma < \sigma^* < \sigma_0$ we have the desired resonance condition. To show that the modified potential satisfies the estimate $V'(x) > cx^2$, in addition to the resonance conditions (3), it suffices to show that eq.(7) holds with $\sigma \approx x^2$ – more precisely, with some σ in $cx_{2n}^2 < \sigma < Cx_{2n}^2$, with c, C independent of n. Indeed, then for any , say, positive, x we have $x_{2n} \leq x \leq x_{2n+2}$. Then $\sigma > cx_{2n}^2 = c(\frac{x_{2n}}{x})^2 x^2 \geq c(\frac{x_{2n}}{x_{2n+2}})^2 x^2 \geq c_1 x^2$, using (5e) in the last inequality. It remains thus to prove that (7) holds with such σ; we do so by estimating both sides of (7) in (8) and (9) below.

Lemma 2. *With the modification of the potential on the interval I_{2n} described above there exist constants $C > c > 0$ independent of n and x_0 such that for all $0 < \sigma \leq \sigma_0$ the times $\tau(\sigma)$ of one–way trips (for $p = 1$) of the solution starting at $(-x_{2n+1}, 0)$ satisfy the estimates*

$$\sqrt{a}(2(\sigma-1)^{-\frac{1}{2}} - \sqrt{2}(\sigma_0 - 1)^{-\frac{1}{2}}) > \tau(\sigma) - \tau(\sigma_0) > \sqrt{a}((\sigma-1)^{-\frac{1}{2}} - \sqrt{2}(\sigma_0-1)^{-\frac{1}{2}}), \tag{8}$$

where $a = x_{2n+2} - x_{2n}$ and

$$\tau(\sigma_0) < Kx_{2n}^{-1}, \tag{9}$$

for some K independent of n. Furthermore, there exists a C^∞– smoothing of the potential which still satisfies the above estimates.

The lemma is proven in section 3.6 below.

3.4. End of proof of the theorem.

To verify the inequality (7) it suffices to make sure that

$$\text{the lower bound on } \tau(\sigma) - \tau(\sigma_0) > \text{upper bound on } 2\tau(\sigma)\tau(\sigma_0);$$

substituting the bounds from (8) and (9), this amounts to

$$\sqrt{a}((\sigma-1)^{-\frac{1}{2}} - \sqrt{2}\sigma_0 - 1^{-\frac{1}{2}}) > 2Kx_{2n}^{-1}(Kx_{2n}^{-1} + \sqrt{a}(2(\sigma-1)^{-\frac{1}{2}} - \sqrt{2}(\sigma_0-1)^{-\frac{1}{2}})).$$

This inequality reduces to an equivalent one

$$(\sigma-1)^{-\frac{1}{2}}(1 - 4Kx_{2n}^{-1}) > 2K^2 a^{-\frac{1}{2}} x_{2n}^{-2} + \sqrt{2}(\sigma_0-1)^{-\frac{1}{2}} - 2\sqrt{2}Kx_{2n}^{-1}(\sigma_0-1)^{-\frac{1}{2}}. \tag{10}$$

The first term in the right–hand side in (10) is the leading one, and is on the order of x_{2n}^{-1}, so that the last inequality should be satisfied with a choice of $\sigma \approx x^2$. To make this precise, we use the estimates $a \equiv x_{2n+2} - x_{2n} > cx_{2n}^{-2}$ and $\sigma_0 = \frac{b}{a} > cx_{2n}^3$ from lemma 1 (eq. (4), (5d), (5e)) to conclude that the right–hand side of (10) is exceeded by Cx_{2n}^{-1}. Taking x_0 so large that the coefficient $1 - 4Kx_{2n}^{-1} > \frac{1}{2}$, we conclude that for some constant C independent of n, (10) and thus (6) hold if $0 < \sigma - 1 \leq Cx_{2n}^2$. This proves that Littlewood's counterexample can be implemented with the quadratic lower bound: $cx^2 < V'$, as explained in the paragraph preceding the statement of Lemma 2. The upper estimate $V' < Cx^3$ is obvious from the fact that the slope of x^4 is not more than doubled in our construction. We emphasize that the choice of x_0 was made independently of n. To complete the proof of the theorem, it remains to observe that the estimates $cx^4 < V(x) < Cx^4$ and $ct^{\frac{1}{3}} < (\dot{x}^2 + x^4)^{\frac{1}{4}} < ct^{\frac{1}{3}}$ follow directly from the above estimates.

3.5. Proof of Lemma 1.

We prove (5a); the other inequalities follow trivially from this one and from (4). Given any $x_k > 0$, the recurrence relation defines the next positive value x_{k+1} uniquely; furthermore, $x_k < x_{k+1}$. The recurrence relation (4) gives $x_{k+1}^4(1 - x_{k+1}^{-3}) = x_k^4(1 + x_k^{-3})$, and thus

$$1 < \left(\frac{x_{k+1}}{x_k}\right)^4 = \frac{1 + x_k^{-3}}{1 - x_{k+1}^{-3}} < \frac{1 + x_0^{-3}}{1 - x_0^{-3}} \equiv r^4;$$

using (4) again, we obtain a homogeneous expression in x_k, x_{k+1} for the difference

$$x_{k+1}^3 - x_k^3 = \frac{(x_k + x_{k+1})(x_k^2 + x_k x_{k+1} + x_{k+1}^2)}{(x_k^3 + x_k^2 x_{k+1} + x_k x_{k+1}^2 + x_{k+1}^3)} = \frac{(1 + \rho_k)(1 + \rho_k + \rho_k^2)}{(1 + \rho_k + \rho_k^2 + \rho_k^3)} \equiv Q(\rho_k).$$

Since $Q(\rho)$ is a decreasing function for $\rho > 1$, and since $1 < \rho_k < r$ by the first estimate, we conclude that $\frac{3}{2} = Q(1) > x_{k+1}^3 - x_k^3 > Q(r) > 1$. Adding these inequalities for $k = 0,\ 1, \ldots,\ n - 1$, we obtain

$$\frac{3}{2}n > x_n^3 - x_0^3 > Q(r)n,$$

which proves (5a). All other estimates (5b–e) follow easily; actually, (5e) has been proven already.

3.6. Proof of lemma 2.

To prove estimate (8) we note that the potentials V_σ and V_{σ_0} differ only on the interval I_{2n} and thus $\tau(\sigma) - \tau(\sigma_0)$ equals the difference $T(\sigma) - T(\sigma_0)$ of the times it takes the solution to cross the interval I_{2n} once. It is easy (but a little messy) to compute $T(\sigma)$ explicitly; instead of doing that we write $T(\sigma) = T_1(\sigma) + T_2(\sigma)$ as the sum of the times it takes for the solution to cross the first and the second halves of I_{2n}, and observe that $T_1(\sigma) < T_2(\sigma)$ (if $0 < \sigma < \sigma_0$, which we assume throughout), so that $T_2(\sigma) < T(\sigma) < 2T_2(\sigma)$, and only T_2 has to be estimated. One computes easily that

$$T_2(\sigma) = \sqrt{a}(\sigma - 1)^{-\frac{1}{2}}, \tag{11}$$

and $T(\sigma_0) = \sqrt{2a}(\sigma_0 - 1)^{-\frac{1}{2}}$. The last three estimates yield (8) at once.

Validity of the estimate (9) is suggested by the fact that it holds for the unmodified potential $U_0(x) = x^4 - x$, as one can check by estimating the period of one oscillation given by the derivative $\frac{\partial A(E)}{\partial E}$ of the area of the level curve with respect to the energy; however, we have to make sure that the cumulative effect of the changes to the potential on successive intervals I_k does not violate this asymptototic behavior. This can be done following Littlewood, by adding up the times the solution spends in each of the intervals I_k; this is done in the appendix. An alternative, slightly shorter method is to show that the ratios of these times to the times a solution in the unmodified potential are all bounded uniformly in n. We give yet another much shorter method; it is based on a comparison idea, sketched in figure 3. By the definition, $\tau(\sigma_0)$ is the time of one one–way trip (when $p = 1$) in the potential $V(x) = V_{\sigma_0}(x)$ that has been modified on the intervals I_k, $k < 2n$, and is linear: $V' = \sigma_0$ on I_{2n}. For $p(t) = 1$ we have $\frac{1}{2}\dot{x}^2 + U(x) = \text{const.}$, where $U(x) = V(x) - x$, and

$$\tau(\sigma_0) = \int_{-x_{2n+1}}^{x_{2n+2}} \frac{dx}{\sqrt{U(x_{2n+2}) - U(x)}}.$$

To estimate this integral, we break it up into the sum

$$\tau(\sigma_0) = \left(\int_{-x_{2n+1}}^{-x_{2n-1}} + \int_{x_{2n}}^{x_{2n+2}} + \int_{-x_{2n-1}}^{x_{2n}} \right) \frac{dx}{\sqrt{U(x_{2n+2}) - U(x)}}$$

of the times of crossing the two extreme intervals I_{2n-1} and I_{2n} and the rest, figure 3.

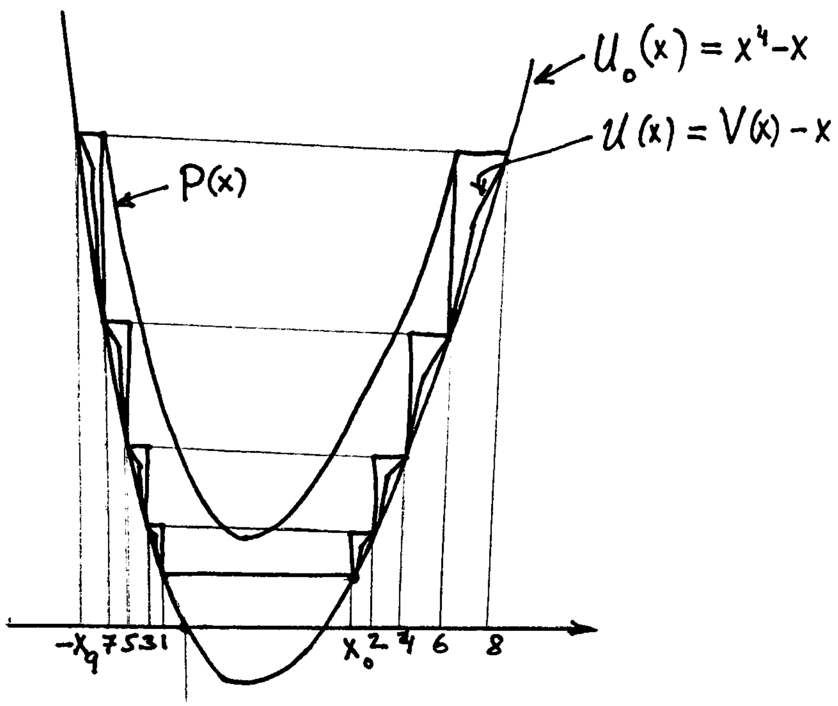

Figure 3. Proof of lemma 2.

The second integral was estimated in (11), with the resulting bound Cx_{2n}^{-2}, using Lemma 1. The first integral is estimated in the same way; the estimate depends only on the inductive assumption $\frac{V'}{x} > cx$ which is valid on I_{2n-1} and on the length of that interval. To bound the integral over the middle interval we introduce the auxiliary potential $P(x) = U_0(x) + U(x_{2n+2}) - U(x_{2n})$, figure 3, where $U_0 = V_0(x) - x \equiv x^4 - x$, and note that $P(x) > U(x)$ on $-x_{2n-1} < x < x_{2n}$, as follows from the fact that the jumps $U(x_{k+2}) - U(x_k)$ are increasing with k, see figure 3. Comparison with the new potential P gives

$$\int_{-x_{2n-1}}^{x_{2n}} \frac{dx}{\sqrt{U(x_{2n+2}) - U(x)}} < \int_{-x_{2n-1}}^{x_{2n}} \frac{dx}{\sqrt{U_0(x_{2n+2}) - P(x)}} = \int_{-x_{2n-1}}^{x_{2n}} \frac{dx}{\sqrt{U_0(x_{2n}) - U_0(x)}};$$

the last integral represents the time of one swing in the potential $U_0(x) = x^4 - x$ (figure 3), and thus is bounded by $\frac{c}{x_{2n}}$, as a simple comparison with the motion in a quartic potential shows. It remains to observe that the potential can be smoothed to a C^∞ function in arbitrarily small neighborhoods of the points where the slope of V jumps in such a way that the effect on the times τ is arbitrarily small.

This completes the proof of lemma 2.

3.7. Appendix: an alternative proof of the estimate on $\tau(\sigma)$.

This is essentially the original proof of Littlewood. To estimate $\tau(\sigma)$, we add up the contributions of each subinterval between $\pm x_k$ and $\pm x_{k+2}$; the integrals over the two extreme intervals I_{2n} and I_{2n-1} are bounded by Cx_{2n}^{-2}, as shown above, and thus are insignificant, with the same result for the middle interval $[-x_1, x_0]$; the main contribution therefore comes from the remaining sum. Breaking up the integral in the way just indicated, we obtain

$$\tau(\sigma_0) = \left(\int_{-x_{2n+1}}^{-x_{2n-1}} + \int_{x_{2n}}^{x_{2n+2}} + \int_{-x_1}^{x_0} + \sum_{k=1}^{n-1} \int_{-x_{2k+1}}^{-x_{2k-1}} + \sum_{k=0}^{n-1} \int_{x_{2k}}^{x_{2k+2}} \right) \frac{dx}{\sqrt{U(x_{2n+2}) - U(x)}},$$

and it remains to prove the bound (9) on the remaining two sums; we do it for the first one. We replace each integrand by its maximum over the interval of integration, and estimate the resulting greater sum

$$\sum_{k=1}^{n-1} \frac{x_{2k+1} - x_{2k-1}}{\sqrt{U(x_{2n+2}) - U(x_{2k-1})}} + \sum_{k=0}^{n-1} \frac{x_{2k+2} - x_{2k}}{\sqrt{U(x_{2n+2}) - U(x_{2k})}}. \tag{12}$$

We replaced $U(x)$ by its maximum on each interval I_k, using the fact that U is increasing for $x > 0$ and decreasing for $x < 0$. The numerators of each summand in (12) are bounded by $Ck^{-\frac{2}{3}}$, according to lemma 1. To estimate the denominators we first note that $U(x_{2k}) = x_{2k}^4 - x_{2k}$, while $U(x_{2k+1}) = x_{2k+1}^4 + x_{2k+1}$. Adding up the recurrence relations for the sequence $\{x_k\}$, we obtain the expressions for the denominators in (12) as the sums (we abbreviate $2n+2 = m$):

$$U(x_m) - U(x_{2k-1}) = (x_{2n+1}^4 + x_{2n+1}) - (x_{2k+1}^4 + x_{2k+1}) = 2 \sum_{j=2k+2}^{2n+1} x_j, \tag{13a}$$

$$U(x_m) - U(x_{2k}) = (x_m^4 - x_m) - (x_{2k}^4 - x_{2k}) = 2\sum_{j=2k}^{2n+1} x_j. \qquad (13b)$$

We bound these sums from below using the estimates $x_k \geq ck^{\frac{1}{3}}$ from lemma 1, together with the inequality $\sum_{j=l}^{m} j^{\frac{1}{3}} > \frac{3}{4}m^{\frac{4}{3}}(1-(\frac{l}{m})^{\frac{4}{3}})$ which one obtains by observing that it is an upper Riemann sum for the integral $m^{\frac{4}{3}}\int_{\frac{l}{m}}^{1} x^{\frac{1}{3}}dx$. Substitution of these estimates in (13a) gives

$$\sum_{j=2k+2}^{2n+1} x_j > C(2n+1)^{\frac{4}{3}}(1-(\frac{2k+2}{2n+1})^{\frac{4}{3}});$$

using this in the denominator of the first sum in (12) results in the upper bound for that sum:

$$C\sum_{k=1}^{n-1}(k+1)^{-\frac{2}{3}}(2n+1)^{-\frac{2}{3}}(1-(\frac{2k+2}{2n+1})^{\frac{4}{3}})^{-\frac{1}{2}};$$

letting $\frac{k+1}{2n+1} = t_k$ and $\Delta t_k \equiv \frac{1}{2n+1}$, we note that the above sum is less than

$$(2n+1)^{-\frac{2}{3}} \sum_{t_k=\frac{2}{2n+1}}^{\frac{n}{2n+1}} t_k^{-\frac{2}{3}}(1-(2t_k)^{\frac{4}{3}})^{-\frac{1}{2}}\Delta t_k < 2(2n+1)^{-\frac{1}{3}}\int_0^{\frac{1}{2}} t^{-\frac{2}{3}}(1-(2t)^{\frac{4}{3}})^{-\frac{1}{2}}dt.$$

Since $(2n+1)^{-\frac{1}{3}} < Cx_k^{-1}$, this estimate gives the desired bound on the first sum in (12). The second sum is estimated in the same way. This proves lemma 2 (again).

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References.

[1] S. Aubry, P. Y. LeDaeron, The discrete Frenkel–Kontorova model and its extensions I: Exact results for the ground states. Physica 8D (1983), pp. 381–422.

[2] G.D. Birkhoff, Dynamical Systems, *AMS Colloq. Publ.*, vol. IX, 1966, pp. 165-169.

[3] R. Dieckerhoff and E. Zehnder, An "a priori" estimate for oscillatory equation. *Dyn. Systems and Bifurcations*, Groningen, 1984. LNM, 1125 Springer. Berlin-New York 1985, pp.9-14.

[4] R. Dieckerhoff and E. Zehnder, Boundedness of solution via the twist–theorem. Ann. Scuola Norm. Sup. Pisa 14(1) 1987, pp. 79–75.

[5] J. Franks, Generalization of the Poincaré-Birkhoff Theorem, to appear in *Annals of Math.*

[6] P. Hartman, On boundary value problems for superlinear second order differential equations, *JDE* 26 (1977), pp.37-53.

[7] M. R. Herman, Sur Les Courbes Invariantes Par Des Diffeomorphismes De L'Anneau, vol. 1, Asterisque 103–104, 1983 and vol. 2, Asterisque 144, 1986.

[8] H. Jacobowitz, R.A. Struble, Periodic solutions of $x'' + f(x,t) = 0$ via the Poincaré-Birkhoff theorem, *J. Diff. Eq.* 20 (1976), no. 1, 37-52. Corrigendum: The existence of the second fixed point: a correction to "Periodic solutions..." above, J.D.E. 25 (1977), no. 1, 148-149.

[9] A. Katok, Some remarks on the Birkhoff and Mather twist theorems. Erg. Th. Dyn. Sys. 2 (1982), 183–194.

[10] J.E. Littlewood, Unbounded Solutions of an equation $\ddot{y}+g(y) = p(t)$, with $p(t)$ periodic and bounded, and $\frac{g(y)}{y} \to \infty$ as $y \to \pm\infty$. J. Lond. Math. Soc., 41 (1966), 497–507.

[11] M. Levi, KAM theory for particles in periodic potentials, to appear in Erg. Th. & Dyn. Sys.

[12] G.R. Morris, A case of boundedness in Littlewood's problem on oscillatory differential equation, *Bull. Austr. Math. Soc.* 14 (1976), 71-93.

[13] J.N. Mather, Existence of quasi–periodic orbits for twist homeomorphisms of the annulus. Topology 21 (1982), pp. 457–476.

[14] J.K. Moser, On invariant curves of area-preserving mappings of an annulus. *Nachr. Acad. Wiss. Göttingen Math-Phys.* Kl II (1962), 1-20.

[15] J.K. Moser, Break–down of stability. Lect. notes in Phys., 247, J.M. Jowett, M. Month and S. Turner, eds, Springer, Berlin, New York, 1986, pp. 492–518.

[16] J.K. Moser, Monotone twist mappings and the calculus of variations. To appear in Erg. Th. & Dyn. Sys.

[17] J.K. Moser, A Stability Theorem for Minimal Foliations on a Torus. Erg. Th. & Dyn. Sys. (1988)8*, 251–281.

[18] H. Rüssmann, Über invariante Kurven differenzierbarer Abbildungen eines Kreisringes, *Nachr. Akad. Wiss., Göttingen II, Math. Phys. Kl., 67–105, 1970.*