Wronskian = Angular Momentum; Abel's Formula = Newton's Second Law

MATHEMATICAL CURIOSITIES

By Mark Levi

Figure 1 depicts a point mass $m=1$ moving in a plane, pulled into the origin by a Hookean spring and subject to friction that is linear in velocity. Position $z \in \mathbb{R}^2$ obeys Newton's second law,

$$\ddot{z} = -qz - p\dot{z}. \qquad (1)$$

We allow q and p to depend on time (so that "Hooke's constant" $q=q(t)$ is constant only in z but not necessarily in t). Taking the two-dimensional cross product with z yields the evolution of the angular momentum $L = z \times \dot{z}$:

$$\frac{d}{dt}(z \times \dot{z}) = -p\,(z \times \dot{z}). \qquad (2)$$

On the other hand, the coordinates of $z=(x,y)$ satisfy the same ordinary differential equation

$$\ddot{u} + p\dot{u} + qu = 0, \qquad (3)$$

and we recognize the angular momentum $L = z \times \dot{z} = x\dot{y} - \dot{x}y \overset{def}{=} W[x,y]$ as these solutions' Wronskian! Newton's law (2) thus becomes

$$\frac{d}{dt}W = -pW, \qquad (4)$$

which is Abel's formula for the Wronskian. This concludes the justification of the claims made in the article's title, where "$=$" stands for "a special case of." Complexification—i.e., going from (3) to (1)—revealed something not seen in one space dimension.

A Logical Question

An entirely different way to understand (4) is to observe that the divergence of any linear vector field is the logarithmic derivative of the area of a region carried by the field (there is no need to consider infinitesimal areas for linear flows). But the divergence of the vector field in the phase plane of (3) is $-p$, and thus the area W of the parallelogram generated by two solution vectors satisfies $\dot{W}/W = -p$, as in (4).

The two aforementioned arguments, which are entirely different, lead to the same conclusion (4). Are these arguments homotopic? This is an interesting question for logicians.

A Hidden Symmetry

Here is another puzzling connection between Abel's theorem and Noether's theorem on conserved quantities. Note that (1) is invariant under rotations; Noether's theorem applies in the conservative case ($p \equiv 0$) and guarantees conservation of the angular momentum, $L = \text{const.}$, thus implying a special case of Liouville's theorem: $W = \text{const}$. One could thus say that the cause is the symmetry hidden in (3) but revealed in (1).

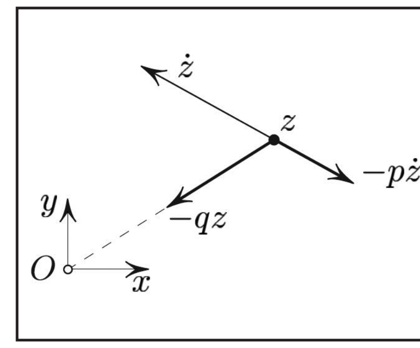

Figure 1. *Both coordinates of z satisfy (3). Here, $p=p(t)$, $q=q(t)$. Figure courtesy of Mark Levi.*

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