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Non-holonomic systems as singular limits for rapid oscillations

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Abstract. In this paper, we point out a close relationship between two standard classical problems in mechanics which have coexisted in textbooks for many decades: (1) the pendulum whose suspension point executes fast periodic motion along a given curve; and (2) the skate (known also as the Prytz planimeter, or the 'bicycle'). More generally, we deal with dynamical systems subjected to rapidly oscillating forcing. Examples include: charged particles in rapidly oscillating electromagnetic fields, in particular the Paul trap; particles in an acoustic wave; a bead sliding on a rapidly vibrating hoop. It turns out that the averaged systems of such kind are approximated by a non-holonomic system. The holonomy turns out to have a transparent geometrical or physical interpretation. For the example of a particle in an acoustic wave the holonomy is directly proportional to the speed of the vibration-induced drift.

1. *The Prytz planimeter*

In this paper, we describe a close connection between two different well-known mechanical systems: the pendulum with vibrating suspension and the so-called Prytz planimeter, also known as the Chaplygin skate. We begin with a brief description of the planimeter.

The Prytz planimeter is a mechanical device, invented by Holger Prytz in about 1875, to measure areas, as mentioned in [7] (see also references therein, for example [16] and [14]).

The Prytz planimeter is sketched in Figure 1. The hatchet-shaped slider end is placed on paper, so that it rests on the slider point B, while the tracer point A is made to trace out the boundary of a domain D, returning to its point of origin. Throughout the motion the slider B moves without sideslip: the velocity of B is aligned with the rod. (The same effect would be achieved by replacing the hatchet by a wheel). The angle between the new and the old directions of AB gives the approximate area of D according to Theorem 1.

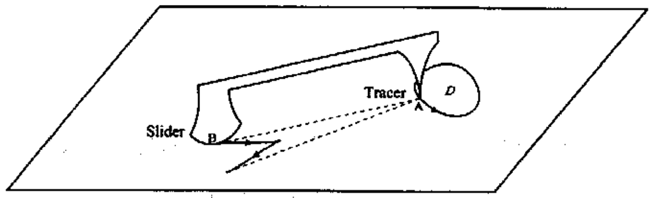

FIGURE 1. Schematics of the Prytz planimeter.

Equations of motion. Let θ be the angle formed by the rod AB with the positive x-axis, and let $\mathbf{z}(t) = \mathbf{z}(t+1)$ be a parametrization of the closed path K traced out by the tracer A. The no-slip constraint of the velocity of the slider B amounts to

$$\dot{\theta} = -\dot{\mathbf{z}} \wedge e^{i\theta} \tag{1}$$

where $\wedge$ denotes the scalar cross product. Indeed, the no-slip constraint gives the alignment of the velocity of B and the rod AB: $(d/dt)(\mathbf{z} + e^{i\theta}) \wedge e^{i\theta} = 0$. Hence $\dot{\mathbf{z}} \wedge e^{i\theta} + \dot{\theta}(ie^{i\theta}) \wedge e^{i\theta} = 0$. Using $(ie^{i\theta}) \wedge e^{i\theta} = -1$ we obtain equation (1). The latter, in the notation $\mathbf{e} = e^{i\theta}$, is equivalent to

$$\dot{\mathbf{e}} = -i(\dot{\mathbf{z}} \wedge \mathbf{e})\mathbf{e}, \tag{2}$$

as one easily checks.

The non-integrable distribution. The position of the system is fully determined by (x, y, θ), where, using the complex notation, $(x, y) \equiv x + iy = \mathbf{z}$. For completeness, we recall the well-known fact that the no-sideslip constraint of the skate is non-integrable, i.e. does not amount to a constraint of a particle in configuration space to a submanifold. Indeed, the no-sideslip constraint on the slider is given by the condition

$$\omega \equiv d\theta + \sin\theta\, dx - \cos\theta\, dy = 0, \tag{3}$$

as follows from equation (1). Since $d\omega \wedge \omega = dx \wedge dy \wedge d\theta \neq 0$, the field of planes (3) is non-integrable, by a theorem of Frobenius, i.e. possesses no tangent surface. Thus the holonomy is non-zero: if $\mathbf{z}$ describes a closed path, $(\mathbf{z}, \theta)$ need not. When the curve is 'small', the asymptotic expression for the change in θ has a nice geometrical interpretation. To make this precise, let the path $\mathbf{z}(t)$ be a homothetic rescaling of a fixed curve $\mathbf{Z}(t)$:

$$\mathbf{z}(t) = \varepsilon\mathbf{Z}(t),$$

so that equation (1) takes form

$$\dot{\theta} = -\varepsilon\dot{\mathbf{Z}} \wedge e^{i\theta} = \varepsilon\dot{\mathbf{Z}} \cdot (-ie^{i\theta}), \tag{4}$$

with $\cdot$ denoting the standard dot product. We will actually look at the more general system

$$\dot{\theta} = \varepsilon\dot{\mathbf{Z}}(t) \cdot \mathbf{F}(\theta), \tag{5}$$

where both $\mathbf{Z} : \mathbb{R} \to \mathbb{R}^2$ and $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ are smooth and periodic, with $\mathbf{Z}(t+1) = \mathbf{Z}(t)$. This system includes equation (1) as a special case. The following theorem is a slight generalization of the theorem due to Prytz and others [7].

THEOREM 1. *Let $\theta = \theta(t)$ be a solution of (5). Under the conditions stated above,*

$$\theta(t_0 + 1) - \theta(t_0) = \varepsilon^2 \mathcal{A}_f + \varepsilon^3 \mathcal{B}_{f,\dot f}(\theta, t_0) + O(\varepsilon^4), \tag{6}$$

where

$$\mathcal{A}_f = \mathcal{A}_f(\theta) = \int_0^1 f' \dot f \, dt, \quad f(t, \theta) = \mathbf{Z} \cdot \mathbf{F}, \tag{7}$$

and

$$\mathcal{B}_{f,g}(\theta, t_0) = \int_0^1 \left(\tfrac{1}{2} f^2 g'' - (f'(t, \theta) - f'(t_0, \theta)) f g'\right) dt. \tag{8}$$

We introduce, for future reference, one more quantity† associated with the planimeter:

$$\mathcal{C}(\theta, \tau) = \frac{1}{2} \frac{d}{d\theta} (\dot f)^2, \quad f(\tau, \theta) = \mathbf{Z} \cdot \mathbf{F}. \tag{9}$$

Geometrical interpretation of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the special case of $\mathbf{F} = -ie^{i\theta} = \langle \sin\theta, \cos\theta \rangle$.

- $\mathcal{A}_f$ is the area enclosed by the path of $\mathbf{Z}(t)$. Indeed, $f' \equiv \mathbf{Z} \cdot \mathbf{F}' = X$ and $f \equiv \mathbf{Z} \cdot \mathbf{F} = Y$ are precisely the coordinates of $\mathbf{Z}$ in the (right-handed) orthonormal frame $(\mathbf{F}', \mathbf{F})$. Thus $\mathcal{A}_f = \int_0^1 f' \dot f \, dt = \oint X \, dY =$ area inside $\mathbf{Z}$, as claimed‡.
- $\mathcal{B}_{f,\dot f}(\theta, t_0) = \bar x \mathcal{A}_f$, where $\bar x$ is the distance from the centroid of the domain $\mathbf{D}$ enclosed by the path of $\mathbf{Z}$ to the line through $\mathbf{Z}(t_0)$ in the direction $-\mathbf{F}(t_0)$. In other words, $\mathcal{B}_{f,\dot f}(\theta, t_0)$ is the torque around the axis through $\mathbf{Z}(t_0)$ in the direction $-\mathbf{F}(t_0)$ of the uniformly distributed force applied to the lamina $\mathbf{D}$, perpendicular to the lamina. Indeed, $f'' = -f$ and (8) with $g = \dot f$ yields

$$\mathcal{B}_{f,\dot f} = \int_0^1 \left(-\tfrac{1}{2} f^2 \dot f - (f' - f_0') f \dot f'\right) dt$$
$$= -\oint (X - X_0) Y \, dX = -\iint_{\mathbf{D}} (X - X_0) \, dX \, dY = \bar x \mathcal{A}_f, \tag{10}$$

  as claimed.
- $\mathcal{C}(\theta, \tau)$ is *the centrifugal acceleration of the slider* and, in particular, $\mathcal{C}(\theta, \tau) = kv^2$, where $k = k(\theta, \tau)$ is the curvature of the slider's path, while $v = v(\theta, \tau)$ is its speed. Indeed, $\mathcal{C}(\theta, t) = \dot f \dot f' = (\dot{\mathbf{Z}}(t) \cdot \mathbf{F}(\theta))(\dot{\mathbf{Z}}(t) \cdot \mathbf{F}'(\varphi))$. Now the first term $\dot{\mathbf{Z}}(t) \cdot \mathbf{F}(\theta)$ in the last product is the angular velocity of the rod and hence of the velocity vector of the slider point, while the second term $\dot{\mathbf{Z}}(t) \cdot \mathbf{F}'(\varphi)$ is the speed of the slider. The product of the two gives the centrifugal acceleration, as claimed.

As a side remark, we state a very nice observation due to Foote [7], followed by a short alternative proof.

THEOREM 2. (Foote) *For any closed curve $\mathbf{z}(t)$ in the plane, the holonomy map $f := e^{i\theta_0} \mapsto e^{i\theta_1}$ of equation (1) is the restriction to the unit circle of a Möbius mapping*

$$T := z \mapsto \frac{z - a}{1 - \bar a z}, \tag{11}$$

where $a \in \mathbb{C}$ is a functional of $\mathbf{z}(\cdot)$.

† The notation $\mathcal{C}$ is suggested by the physical origin of this term as the force of *constraint* (in an associated non-holonomic system).
‡ Here $' = d/d\theta$ and $\dot{} = d/dt$.

We give a very short alternative proof of this theorem, based on the observation that the infinitesimal generator (i.e. the vector field on the circle given by the turning of the planimeter's rod) of the holonomy map belongs to the Lie algebra of the Lie group of Möbius transformations.

Proof. All Möbius transformations of the form (11) leave the unit circle $|w| = 1$ invariant. These transformations, restricted to the unit circle, form a Lie group G. To prove the theorem it suffices to show that the Lie algebra $\mathcal{G}$ of G contains all vectorfields of the form (2). To that end choose a parametrized curve T_λ in G given by $T_\lambda(w) = (w - \lambda a)/(1 - \lambda \bar{a} w)$, $|w| = 1$, with an arbitrary $a \in \mathbb{C}$. We have $T_0 = \mathrm{id} \in G$. Consider the tangent vector at identity

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} T_\lambda(w) = -a - w(-\bar{a}w) = (\bar{a}w - a\bar{w})w = 2i(a \wedge w)w, \tag{12}$$

we used $|w| = 1$. This vectorfield coincides with (2) if we set $a = -\frac{1}{2}\dot{z}$. Thus the vectorfields (2) generated by the planimeter lie in the Lie algebra of the Möbius vectorfields, as claimed. □

This completes our discussion of the necessary background. We now proceed to the main point of this paper.

2. *Holonomy arising in the normal form*

In this paper we focus on systems of the form

$$\ddot{\theta} + k\dot{\theta} = \mathbf{a}(t) \cdot \mathbf{F}(\theta), \quad \mathbf{a} : \mathbb{R} \to \mathbb{R}^2, \ \mathbf{F} : \mathbb{R} \to \mathbb{R}^2, \tag{13}$$

where $\mathbf{a}(t) \cdot \mathbf{F} = a_1(t)F_1(\theta) + a_2(t)F_2(\theta)$ and where $k > 0$.

The prime example of such a system is the pendulum, i.e. a segment in the plane with a point mass at one end and with the other end (the hinge), $\mathbf{z}(t) = (x(t), y(t))$, undergoing prescribed periodic motion along a planar curve (the 'twirled' pendulum). With the assumption of linear friction in the hinge and with an appropriate rescaling, the *clockwise* angle θ of the pendulum is governed by equation (13) where $\mathbf{a} = \ddot{\mathbf{z}}(t) - \mathbf{g}$ and $\mathbf{F} = \langle \sin\theta, -\cos\theta \rangle = -ie^{i\theta}$ is the clockwise tangent vector to the circle†:

$$\ddot{\theta} + k\dot{\theta} = \mathbf{a}(t) \cdot \mathbf{F}(\theta) \equiv A(t)\cos(\theta - \phi(t)); \tag{14}$$

here $\mathbf{g} = (0, -g)$ is the gravitational acceleration.

Scaling assumptions. We assume throughout that

$$\mathbf{a}(t) = \varepsilon^{-1}\mathbf{A}\left(\frac{t}{\varepsilon}\right), \quad \varepsilon \ll 1, \tag{15}$$

† Derivation of the equation is as follows Let $\mathbf{a}(t, \varepsilon)$ be the acceleration of the suspension point. In the accelerating frame of the suspension point, the point mass of the pendulum is subject to the d'Alembert force $-\ddot{\mathbf{z}}(t, \varepsilon)$. The tangential component to the unit circle in the direction $-\mathbf{T}$ of increasing θ is $\ddot{\mathbf{z}} \cdot \mathbf{T}(\theta)$. The (only) other tangential forces with nonzero component in the direction $-\mathbf{T}$ are the gravitational $-\mathbf{g} \cdot \mathbf{T}$ and frictional $-k\dot{\theta}$.

where $\mathbf{A} \in C^3(\mathbb{R}, \mathbb{R}^2)$, where $\mathbf{A}(\tau) = \mathbf{A}(\tau + 1)$ is a fixed function. For the case of the pendulum this amounts to the assumption of small-amplitude, large-acceleration vibrations of the pivot. We introduce the non-dimensional velocity $\mathbf{V}$ and the position $\mathbf{Z}$ via

$$\mathbf{V}(\tau) = \int \mathbf{A}(s)\, ds, \quad \overline{\mathbf{V}} = \mathbf{0}, \tag{16}$$

where $\overline{V}$ denotes the average value of V and

$$\mathbf{Z}(\tau) = \int \mathbf{V}(s)\, ds, \quad \overline{\mathbf{Z}} = \mathbf{0}. \tag{17}$$

Remark 1. The case of the pivot oscillating along a straight line segment, $\mathbf{Z} = Z(t)\mathbf{g}$, where $\mathbf{g} \in \mathbb{R}^2$ is constant, is the classical forced pendulum†:

$$\ddot{\theta} + k\dot{\theta} = p(t)\cos\theta. \tag{18}$$

For $k = 0$, the linearization of this is Hill's equation, which is a fundamental dynamical system for many applications and which has been studied extensively; see **[2, 6, 8, 18, 19]** and references therein. The appearance of a non-holonomic constraint in (18) with $k = 0$ has been observed in **[13]**. We show that the relationship extends further in the presence of dissipation $k \neq 0$ and with the pivot following an arbitrary closed path.

2.1. *The geometrical normal form.* The following main theorem shows that the three terms $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ associated with the Prytz planimeter arise in the normal form of the inverted pendulum or for the slightly more general case of equation (13).

THEOREM 3. *There exists a change of variables $\theta = \varphi + \varepsilon g(\varphi, t, \varepsilon)$ with $g(\varphi + 2\pi, t, \varepsilon) = g(\varphi, t + \varepsilon, \varepsilon) = g(\varphi, t, \varepsilon)$, such that equation (13) with $\mathbf{a}(t) = \varepsilon^{-1}\mathbf{A}(t/\varepsilon)$ and $\overline{\mathbf{A}} = \mathbf{0}$, is transformed into*

$$\ddot{\varphi} + k\dot{\varphi} = \overline{\mathcal{C}}_f(\varphi) + \varepsilon\big(k\overline{\mathcal{A}}_f(\varphi) + \overline{\mathcal{B}}_{f,\tilde{f}}(\varphi)\big) + \varepsilon^2 R(\varphi, t, \varepsilon), \tag{19}$$

where $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{B}$ are the quantities associated with the Prytz planimeter and given by equations (9), (7) and (8) respectively, and $\overline{X}$ denotes the time average of X.

The proof of the theorem is given in the Appendix.

It should also be noted that if $\mathbf{a}$ and $\mathbf{F}$ are analytic, then the system can be brought to an exponentially small perturbation of an autonomous system. It is not our goal, however, to consider this higher-order reduction which is well understood in settings more general than ours (see **[5]** and **[15]**, among others).

In the special case of the pendulum, $\mathbf{F} = -ie^{i\theta}$, we have:

- $\mathcal{C}_f(\theta, \tau)$ is the centrifugal acceleration of the slider point of the associated planimeter;
- $\varepsilon\mathcal{A} =$ (area enclosed by path of $\mathbf{z}$)/(period of $\mathbf{a}$) $= O(\varepsilon)$, independent of θ.

We note the difference between the terms $\mathcal{B}_{f,\dot{f}}(\varphi)$ and $\mathcal{B}_{f,\tilde{f}}(\varphi)$ in equations (8) and (19), respectively. The latter thus no longer has the mechanical interpretation of the torque.

We do not address here the interesting question of determining the algebraic correspondence between the two systems to all orders.

† For a discussion in the physical literature, see, for example, **[1, 4, 9–11]**.

Remark 2. The area term $\mathcal{A}$ in the averaged equation (19) is absent in the conservative case $k = 0$.

3. *Applications*

3.1. *A bead on a rigid hoop.*

Consider a point mass sliding on the rigid hoop with friction proportional to the speed of sliding along the hoop. No additional forces, apart from the constraint to the hoop, act on the bead†.

Adiabatic motion of this system in the conservative case has been studied by Berry and Hannay [3], who showed that as the hoop undergoes one slow revolution around a fixed point, the bead running with speed $O(1)$ around the hoop ends up shifted—compared to its position on a stationary hoop—by the arclength $A/2\pi L$, where A and L are respectively the area and the length of the hoop. In this section, we demonstrate a different holonomy effect in the 'opposite' asymptotic case of a rapidly vibrating hoop *and* in the presence of friction. Again, as in the previous case, we will see an associated non-holonomic system as the singular limit of the original holonomic system.

Equations of motion. Consider a smooth closed curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ parametrized by the arclength s and let this curve move: $\mathbf{r}(s,t) = -\mathbf{z}(t) + \mathbf{r}(s)$. We assume small-amplitude, high-frequency motions: $\mathbf{z}(t,\varepsilon) = \varepsilon\mathbf{Z}(t/\varepsilon)$ (Figure 2). The negative sign is used for convenience later. The hoop thus oscillates in a closed path, undergoing parallel translations by $-\mathbf{z}$. The equations of motion of the bead are

$$\ddot{s} + k\dot{s} = \mathbf{a}\cdot\mathbf{T}(s), \tag{20}$$

where $\mathbf{a}(t) = \ddot{\mathbf{z}}$ and $\mathbf{T} = \mathbf{r}'(s)$ is the unit tangent vector. Indeed, in the hoop's frame the bead is subject to three forces:

- the d'Alembert force $-m(-\ddot{\mathbf{z}}) = m\mathbf{a} = \mathbf{a}$ (we take $m = 1$);
- the frictional force, linear in speed, $-k\dot{s}\mathbf{T}$; and
- the reaction force normal to $\mathbf{T}$.

The projection of these forces onto $\mathbf{T}$ gives $\ddot{s} = -k\dot{s} + \mathbf{a}\cdot\mathbf{T}$, as claimed.

Remark 3. If the hoop is a circle, then the problem reduces to the pendulum with a vibrating suspension.

With $\mathbf{z} = \varepsilon\mathbf{Z}(t/\varepsilon)$, Theorem 3 applies: there exists a transformation $s = \varphi + \varepsilon g(\varphi, t, \varepsilon)$ turning the equation (20) into (19). The averaged equations take the form

$$\ddot{\sigma} + k\sigma = \overline{\kappa^{\perp}u^2} + \varepsilon\big(\kappa\mathcal{A} + \overline{\mathcal{B}_{f,\dot{f}}}\big), \tag{21}$$

where:

- $\kappa^{\perp} = \kappa^{\perp}(\sigma, t)$ is the curvature of the curve normal to the family of translates $\mathbf{r}(\sigma) - \mathbf{z}(t)$;
- $u = \dot{\mathbf{z}} \wedge \mathbf{r}'$ is the normal velocity of the hoop;
- $\mathcal{A}$ is the area enclosed by the curve $\mathbf{z}(t)$; and
- $\kappa(\sigma) = |\mathbf{r}''(\sigma)|$ is the curvature of the hoop as the function of the arclength.

We leave out the simple justification of the above interpretations, mentioning only that the term $\kappa u^{\perp}$ could be guessed from heuristic considerations along the lines of **[12]**.

† The special case of the circular hoop is the forced pendulum.

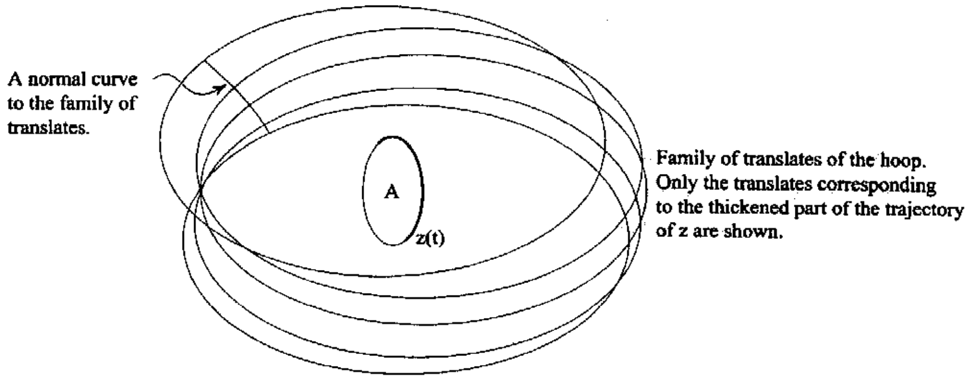

FIGURE 2. The geometrical interpretation of C for the hoop.

3.2. *A particle in an acoustic wave.* Consider a traveling planar compression wave in a medium, with the displacement given by $\bar{a}\cos(x - ct)$. Here x is the coordinate in the direction of propagation of the wave. A suspended particle of mass m subject to a linear drag force obeys $m\ddot{x} + \bar{k}x = \bar{a}\cos(x - ct)$, or dividing by m and renaming the constants:

$$\ddot{x} + kx = a\cos(x - ct). \tag{22}$$

The equation can be rewritten in our standard form with $\mathbf{a} = a\langle -\sin ct, \cos ct\rangle$ and $\mathbf{F} = \langle -\sin x, \cos x\rangle$:

$$\ddot{x} + k\dot{x} = \mathbf{a}(t)\cdot\mathbf{F}(x). \tag{23}$$

Assume that $c^{-1} = \varepsilon$ is small and that $a = \bar{a}/m = O(1/\varepsilon)$, so that the asymptotic assumptions (15) hold.

Thus Theorem 3 applies; the first and the third terms in the normal form on the right-hand side vanish in this special case and the averaged equation simplifies to

$$\ddot{y} + k\dot{y} = \varepsilon k\mathcal{A} + O(\varepsilon^2), \tag{24}$$

where $\varepsilon k\mathcal{A} = ka^2/(2c^3)$. We arrive at a somewhat unexpected result.

COROLLARY 1. *The speed of drift is independent of the coefficient of friction k to the leading order in ε.*

Indeed, each solution of the truncated system $\ddot{y} + k\dot{y} = ka^2/(2c^3)$ approaches exponentially to a linear motion $y = a^2/(2c^3)t +$ constant.

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A. *Appendix. Normal form calculation*

As before, we define $f(\tau, \theta) = (\mathbf{Z}(\tau)\cdot\mathbf{F}(\theta))$, $\dot{f} \equiv \partial f/\partial t$ and $f' \equiv \partial f/\partial\theta$. Equation (13), with the scaling assumption (15), is then

$$\ddot{\theta} + k\dot{\theta} = \varepsilon^{-1}\ddot{f}\left(\frac{t}{\varepsilon}, \theta\right). \tag{A.1}$$

In this Appendix, we show that this equation may be transformed into

$$\ddot{\varphi} + k\dot{\varphi} = -\overline{\dot{f}\dot{f}'} + \varepsilon\left\{k\overline{\dot{f}f'} + \tfrac{1}{2}\overline{f^2\ddot{f}''} - \overline{ff'\ddot{f}'}\right\} + O(\varepsilon^2), \tag{A.2}$$

from which Theorem 3 follows. We assume that the time average of $\ddot{f}$ is zero, but at the end of this Appendix we shall also give the equation that results if this average is not zero.

Let $\tau = t/\varepsilon$ and $\theta = \varepsilon X$. Then (A.1) becomes

$$X'' + \varepsilon k X' = \ddot{f}(\tau, \varepsilon X). \tag{A.3}$$

We make the following transformation, where h is to be determined:

$$X = x + h(\tau, \varepsilon x). \tag{A.4}$$

Substitute (A.4) in (A.3), and relabel x to X to obtain

$$X'' + h_{11} + \varepsilon\{2h_{12}X' + kX' + kh_1 - h_{11}h_2\} + \varepsilon^2\{h_{22}(X')^2 - 2h_2h_{12}X' - kh_1h_2 + h_2^2h_{11}\}$$
$$= \ddot{f} + \varepsilon\{h\ddot{f}' - h_2\ddot{f}\} + \varepsilon^2\left\{\left(\frac{h^2}{2}\right)\ddot{f}'' - hh_2\ddot{f}' + h_2^2\ddot{f}\right\} + O(\varepsilon^3). \tag{A.5}$$

Choose h, periodic in τ and with zero average, so that the $O(1)$ terms are autonomous. The appropriate choice is

$$h(\tau, \varepsilon X) = \int_{\tau_0}^{\tau}\int_{\tau_0}^{\hat{\sigma}} \ddot{f}(\sigma, \varepsilon X)\, d\sigma\, d\hat{\sigma} \tag{A.6}$$

so $h = f$, $h_1 = \dot{f}$, $h_2 = f'$, $h_{11} = \ddot{f}$, $h_{12} = \dot{f}'$ and $h_{22} = f''$. Then (A.5) becomes

$$X'' + \varepsilon\{2\dot{f}'X' + kX' + k\dot{f}\} + \varepsilon^2\{(f'')(X')^2 - 2f'\dot{f}'X' - k\dot{f}f'\}$$
$$= \varepsilon f\ddot{f}' + \varepsilon^2\left\{\left(\frac{f^2}{2}\right)\ddot{f}'' - ff'\ddot{f}'\right\} + O(\varepsilon^3). \tag{A.7}$$

Rewrite as a system by defining $Y = X'$:

$$\begin{aligned} X' &= Y \\ Y' &= \varepsilon\{f\ddot{f}' - 2\dot{f}'X' - kX' - k\dot{f}\} \\ &\quad + \varepsilon^2\left\{\left(\frac{f^2}{2}\right)\ddot{f}'' - ff'\ddot{f}' - (f'')(X')^2 + 2f'\dot{f}'X' + k\dot{f}f'\right\} + O(\varepsilon^3). \end{aligned} \tag{A.8}$$

We now average the $O(\varepsilon)$ terms. Transform X and Y as follows:

$$\begin{aligned} X &= x + \varepsilon u(\tau, \varepsilon x, y), \\ Y &= y + \varepsilon v(\tau, \varepsilon x, y). \end{aligned} \tag{A.9}$$

We substitute (A.9) into (A.8). After some further manipulation, we relabel (x, y) to

(X, Y) and obtain

$$X' = Y + \varepsilon\{v - u_1\} + \varepsilon^2\{-u_2 Y + u_3[v_1 - f\ddot{f}' + 2\dot{f}'Y + kY + kh_1\dot{f}]\} + O(\varepsilon^3) \tag{A.10}$$

$$\begin{aligned} Y' &= \varepsilon\{-v_1 + f\ddot{f}' - 2\dot{f}'Y - kY - k\dot{f}\} \\ &\quad + \varepsilon^2\left\{ -v_2 Y + \left(\frac{f^2}{2}\right)\ddot{f}'' - ff'\ddot{f}' - f''Y^2 + 2f'\dot{f}'Y + k\dot{f}f' \right. \\ &\quad \left. + v_3[v_1 - f\ddot{f}' + 2\dot{f}'Y + kY + k\dot{f}] \right\} + O(\varepsilon^3). \end{aligned} \tag{A.11}$$

We now choose u and v (periodic in τ and with zero average) so that the $O(\varepsilon)$ terms in (A.10) and (A.11) are autonomous. In (A.11), we find the appropriate choice of v to be

$$v(\tau, \varepsilon X, Y) = -2f'Y - kf + \int_{\tau_0}^{\tau} f\ddot{f}' - \overline{f\ddot{f}'}\, d\sigma. \tag{A.12}$$

From (A.10), we determine u to be

$$u(\tau, \varepsilon X, Y) = -2Y\int_{\tau_0}^{\tau} f'\, d\sigma - k\int_{\tau_0}^{\tau} f\, d\sigma + \int_{\tau_0}^{\tau}\int_{\tau_0}^{\sigma} f\ddot{f}' - \overline{f\ddot{f}'}\, d\hat{\sigma}\, d\sigma. \tag{A.13}$$

With these choices of u and v, (A.10) and (A.11) become

$$\begin{aligned} X' &= Y + \varepsilon^2\left\{ -u_2 Y + u_3\left(\overline{\dot{f}\dot{f}'} + kY\right)\right\} + O(\varepsilon^3) \\ Y' &= \varepsilon\left\{ -\overline{\dot{f}\dot{f}'} - kY \right\} \\ &\quad + \varepsilon^2\left\{ -v_2 Y + \left(\frac{f^2}{2}\right)\ddot{f}'' - ff'\ddot{f}' - f''Y^2 + 2f'\dot{f}'Y \right. \\ &\quad \left. + k\dot{f}f' + v_3\left(\overline{\dot{f}\dot{f}'} + kY\right)\right\} + O(\varepsilon^3). \end{aligned} \tag{A.14}$$

We have used $\overline{f} = 0$ and

$$\overline{f\ddot{f}'} = \int_0^1 f\ddot{f}'\, d\sigma = -\int_0^1 \dot{f}\dot{f}'\, d\sigma = -\overline{\dot{f}\dot{f}'}. \tag{A.15}$$

We now make another change of variables to make the $O(\varepsilon^2)$ terms autonomous:

$$\begin{aligned} X &= x + \varepsilon^2 p(\tau, \varepsilon x, y), \\ Y &= y + \varepsilon^2 q(\tau, \varepsilon x, y). \end{aligned} \tag{A.16}$$

Put (A.16) into (A.14) and relabel (X, Y) to (x, y):

$$\begin{aligned} X' &= Y + \varepsilon^2\left\{ q - p_1 - p_3\overline{\ddot{f}} - u_2 Y + u_3\left(\overline{\dot{f}\dot{f}'} + kY\right)\right\} + O(\varepsilon^3) \\ Y' &= \varepsilon\left\{ -\overline{\dot{f}\dot{f}'} - kY \right\} \\ &\quad + \varepsilon^2\left\{ -q_1 - v_2 Y + \left(\frac{f^2}{2}\right)\ddot{f}'' - ff'\ddot{f}' - f''Y^2 + 2f'\dot{f}'Y \right. \\ &\quad \left. + k\dot{f}f' + v_3\left(\overline{\dot{f}\dot{f}'} + kY\right)\right\} + O(\varepsilon^3). \end{aligned} \tag{A.17}$$

Now choose p and q, periodic with zero average, so that the $O(\varepsilon^2)$ terms are autonomous. We will not need the explicit formulas for p and q. We only need to note that the result of this choice will be to leave behind the average of the non-autonomous terms. The resulting system is

$$\begin{aligned} X' &= Y + O(\varepsilon^3) \\ Y' &= \varepsilon\left\{-\overline{\dot{f}\dot{f}'} - kY\right\} + \varepsilon^2\left\{\tfrac{1}{2}\overline{f^2\ddot{f}''} - \overline{ff'\ddot{f}'} + k\overline{\dot{f}f'}\right\} + O(\varepsilon^3). \end{aligned} \tag{A.18}$$

Upon converting this system back to a second-order equation and using $\varphi = \varepsilon X$ and $t = \varepsilon\tau$, we obtain (A.2).

If the time average of $\ddot{f}$ is not zero, the previous calculations can be modified to show that the transformed system is

$$\ddot{\varphi} + k\dot{\varphi} = \varepsilon^{-1}\overline{\ddot{f}} - \overline{\dot{f}\dot{f}'} + \varepsilon\left\{k\overline{\dot{f}f'} + \tfrac{1}{2}\overline{f^2\ddot{f}''} - \overline{ff'\ddot{f}'} - \overline{(f')^2}\,\overline{\ddot{f}}\right\} + O(\varepsilon^2). \tag{A.19}$$

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