Ulam's Ping-pong, Adiabatic Invariants, and Entropy

MATHEMATICAL CURIOSITIES

By Mark Levi

In the early days of quantum mechanics, physicists wondered why Planck's constant is constant — why is the energy/frequency ratio of photons fixed? After all, the atom that emits photons is buffeted by a surrounding electromagnetic field; it is surprising that this buffeting does not change the ratio.

At a Solvay Conference in 1911, Einstein showed that a slightly analogous phenomenon occurs for the mathematical pendulum. If the string's length is changed appreciably but slowly enough (taking a long time), then the ratio changes arbitrarily little. Quantities that behave in such a way are called *adiabatic invariants*.

Einstein derived the adiabatic invariance of the pendulum from the conservation of energy: *the work of pulling the string is spent on lifting the bob and on changing the bob's energy of oscillations*. With some massaging, this statement yields the conclusion that E/ω is an adiabatic invariant.

In addition to Einstein's physical explanation (which I believe can be made rigorous), another one exists that is based on the action-angle variables [1]; we can actually turn it into a geometrical proof that is almost free of formulas.

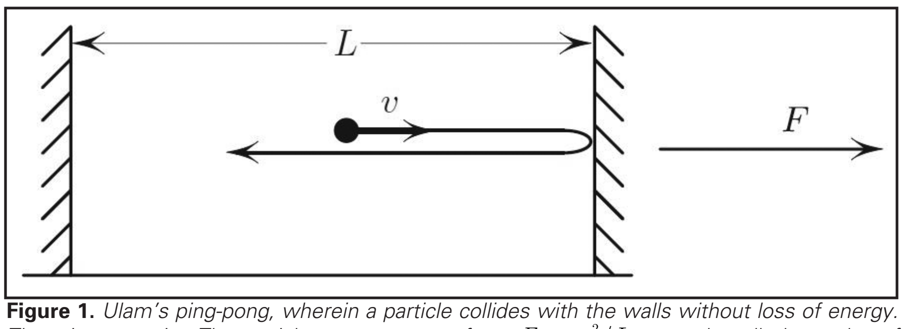

Figure 1. *Ulam's ping-pong, wherein a particle collides with the walls without loss of energy. There is no gravity. The particle exerts average force $F = mv^2/L$ on each wall: the analog of pressure in this single-atom gas.*

The coexistence of two seemingly unrelated explanations of the same effect suggests the need to rise above the maze to see the whole picture at a glance; as far as I know, this remains to be done. Instead, I would like to illustrate Einstein's idea on a simple toy model of Ulam's ping-pong: a particle bouncing between two walls — a baby model of ideal gas (see Figure 1).

Unlike in Figure 1, the wall in Figure 2 moves in slowly. The work spent on pushing the wall adds to the kinetic energy of the "molecule":

$$F dL \approx d\left(\frac{mv^2}{2}\right). \quad (1)$$

Here, F is the averaged force of impacts on the wall in Figure 1 for the fixed wall; the "$\approx$" sign is due to the fact that I replaced the averaged force that the mover applied in Figure 2 with F — associated with the fixed wall in Figure 1. I claim that

$$F = \frac{mv^2}{L}; \quad (2)$$

this is proven at the end.

Substitution of (2) into (1) gives

$$\left(-\frac{mv^2}{L}\right) dL \approx d(mv^2/2). \quad (3)$$

The minus sign is due to the fact that the mover in Figure 2 pushes left, i.e., in the negative direction. Rearranging (3) yields

$$\frac{dL}{L} + \frac{dv}{v} \approx 0 \quad \text{or} \quad d\big(\underbrace{\ln vL}_{\text{entropy}}\big) \approx 0$$

so that $vL \approx$ const.

Here are some remarks:

1. vL is the area of the path in the phase plane when the walls are fixed (see Figure 3).

2. $\ln vL$, the log of the phase area, is the *entropy* of our "gas." It is additive because phase volumes multiply when we add more dimensions. Just like in this simple case, *the entropy of ideal gas is the log of the volume that is enclosed by the energy hypersurface in the phase space*. It is also an adiabatic invariant of the gas.

3. As with the pendulum, the energy/frequency interpretation for Ulam's ping-pong holds: $vL = \frac{E}{\omega}$, where

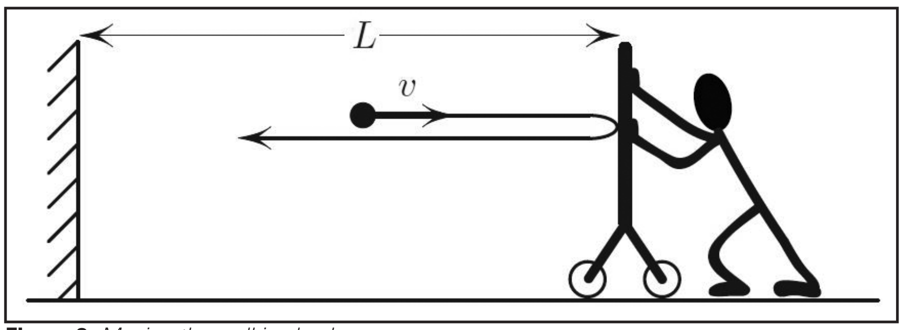

Figure 2. *Moving the wall in slowly.*

$E = v^2/2$ is the energy of unit mass and $\omega = 1/T = 1/(2L/v)$.

4. The adiabatic invariance of vL is the one-dimensional version of the adiabatic equation for ideal gas:

$$pV^{\gamma} = \text{const.}, \quad \text{where} \quad \gamma = \frac{f+2}{f}, \quad (4)$$

Here, f is the number of degrees of freedom; $f = 1$ and $\gamma = 3$ for our "gas." To see this connection, we note that pressure, volume, and temperature (p, V, T) in ideal gas correspond to our F, L, and $mv^2/2$ respectively. Since (4) involves pressure and volume, we also wish to express our adiabatic invariant vL in terms of "pressure" F and "volume" L. Substituting the v from (2) into vL yields

$$vL = \sqrt{FL^3/m},$$

so that $FL^3 \approx$ const. This is the exact counterpart of (4) for $f = 1$.

5. pV^{γ}, or rather its power, has a geometrical interpretation as the volume enclosed by the energy surface in the phase space.

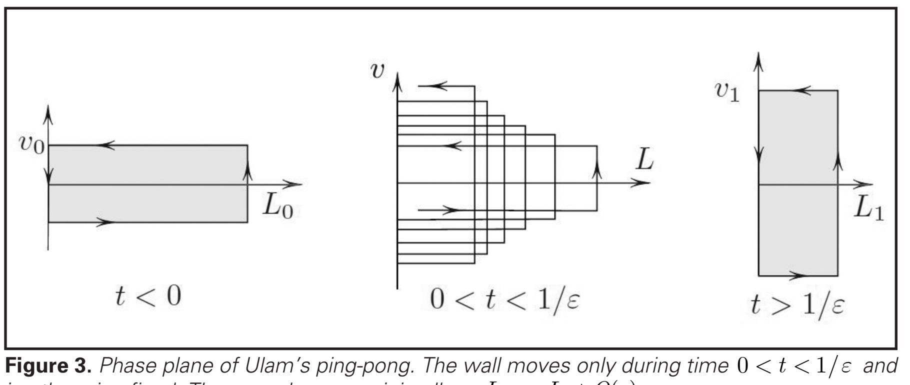

Figure 3. *Phase plane of Ulam's ping-pong. The wall moves only during time $0 < t < 1/\varepsilon$ and is otherwise fixed. The area changes minimally: $v_0 L_0 = v_1 L_1 + O(\varepsilon)$.*

Proof of (2). The momentum in each collision with the right wall in Figure 1 changes from mv to $-mv$, i.e., by $2mv$ at each impact. These impacts are spaced at times $T = 2L/v$ apart. The change of momentum per unit time, i.e., the force, is hence

$$\frac{2mv}{T} = \frac{mv^2}{L},$$

as claimed.

Interestingly, this is exactly the same force as the centripetal force upon a mass m that moves in a circle of radius L with speed v; in fact, we can derive (2) by considering motion with speed v on a circle but one of radius $L/2$; I omit this alternative derivation.

The figures in this article were provided by the author.

References

[1] Arnold, V.I. (1989). *Mathematical methods of classical mechanics*. New York, NY: Springer-Verlag.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.