# Focusing on Nephroids

MATHEMATICAL CURIOSITIES

By Mark Levi

The bright strip in Figure 1 is illuminated by the incoming parallel beam reflected from the inner surface of the cup. The rays do not focus at a single point, as they would if the wall were parabolic. Instead, they "focus" at a curve, with the cusp at the focus of the osculating parabola.

Figure 2 sheds some light on the situation: the density of reflected rays spikes at the envelope, referred to as the *caustic*. This explains the caustic's brightness.

Remarkably, this caustic is an epicycloid — more precisely, the path of a particle on the rim of the wheel rolling without sliding on another wheel, with a 1:2 ratio of radii (see Figure 3). Because of its vaguely kidney-like shape, this epicycloid is referred to as a *nephroid* (kidney$=\nu\varepsilon\varphi\rho o$).

The precise statement is the following. *For every ray from a pencil of parallel rays striking the inside of a circular mirror of radius $R$, the reflected ray is tangent to the nephroid generated by rolling a circle of radius $R/4$ on the stationary circle of radius $R/2$ concentric with the mirror. The cusps of this nephroid lie on the ray passing through the center of the mirror.*

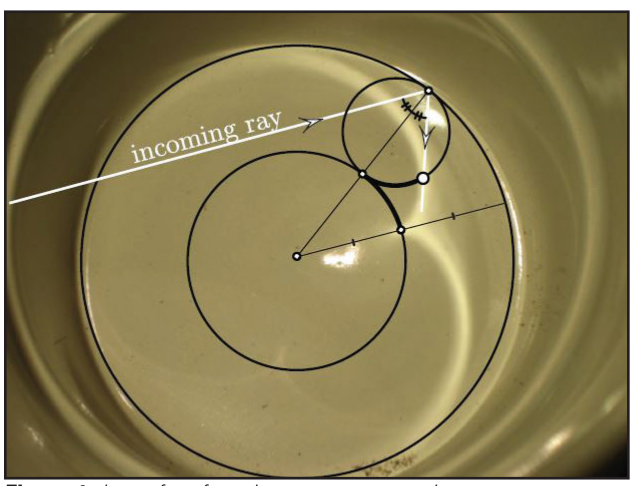


**Figure 1.** *Imperfect focusing creates a caustic.*

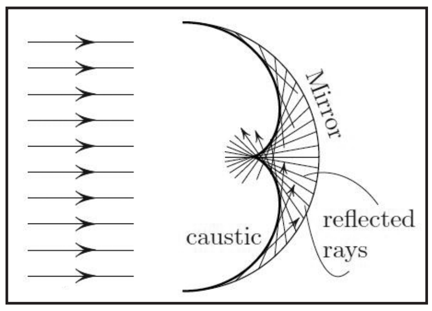


**Figure 2.** *The density of reflected rays spikes at the caustic. This caustic is half of the nephroid in Figure 3.*

To restate this result, imagine walking around the circle with constant angular velocity 1 and twirling a baton in the horizontal plane with angular velocity 2. The envelope of the resulting family of lines is a nephroid.

### Proof of the Claim

Consider the nephroid generated by rolling the circle, as seen in Figures 3 and 4; $T$ is the point tracing out the nephroid. We must prove that $PT$ is the reflected ray, i.e., that

$$\angle APO=\angle OPT, \tag{1}$$

and that $PT$ is tangent to the nephroid.

Note that $\mathrm{arclength}(CS)=\mathrm{arclength}(CT)$ due to the non-slip condition, so that

$$\angle COS=\frac{1}{2}\angle CQT,$$

because the radii are in the 1:2 ratio. In turn, $\angle CQT=2\angle CPT$, since these angles subtend the same arc. This proves (1).

Why is $TP$ tangent to the nephroid? As the smaller circle rolls on the larger one, the velocity $\mathrm{v}_T\perp CT$, because $C$ is the rolling wheel's instantaneous center of rotation. But $TP\perp CT$, since $CP$ is a diameter. Thus, $\mathrm{v}_t \parallel TP$; $TP$ is indeed tangent. □

On a related note, if the source of light lies on the circle, then the resulting caustic is a cardioid, i.e., the epicycloid generated by the circle rolling on another circle of equal radius (see Figure 5). This proof is the same as the one above.

To conclude, here is a small challenge: show that the length of the thick line in Figure 5 is independent of the choice of $B$, namely

$$AB+BT+TC=\frac{8}{3}R, \tag{2}$$

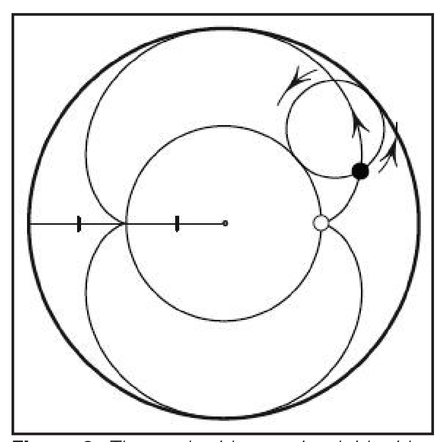

**Figure 3.** *The nephroid: an epicycloid with a* 1:2 *ratio of radii.*

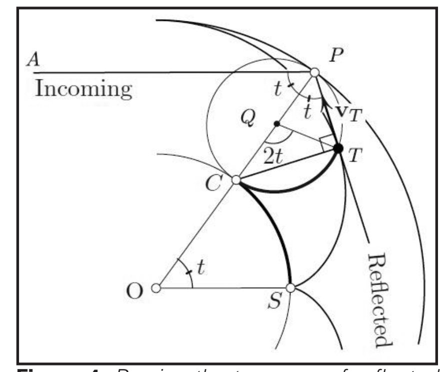


**Figure 4.** *Proving the tangency of reflected rays to the nephroid, which is traced by point $T$ on the rolling wheel. The starting position of $T$ is $S$.*

$R$ being the radius of the mirror. Assuming that (2) holds, setting $B=A$ causes the first two terms to vanish, and we conclude that the cardioid's length is $16R/3$.

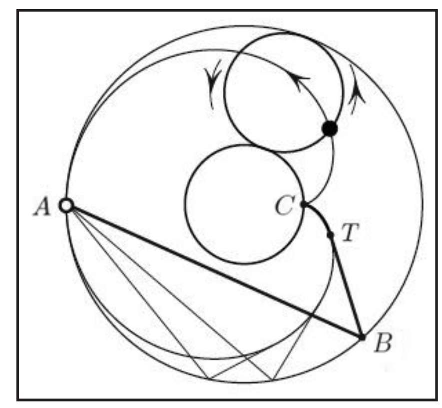


**Figure 5.** *The cardioid—an epicycloid with the* 1:1 *ratio—is the caustic created by the source of light lying on the circular mirror.*

*The author provided the preceding figures.*

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