A Minimalist Minimizes an Integral

MATHEMATICAL CURIOSITIES
By Mark Levi

In this issue we present a solution that is shorter than Johann Bernoulli's famous optics-based idea of minimizing

$$\int_{\gamma OA} F(y)\,ds \quad (1)$$

over smooth curves connecting two given points A and B; here $F(y) > 0$ is a given function and ds is an element of arc length. Bernoulli based his beautiful solution on the equivalence between Fermat's principle and Snell's law.

The following solution, in addition to being shorter, substitutes a mechanical analogy for Bernoulli's optical one—and thus could have been given by Archimedes.

The sum

$$P_N = \sum F(y_k)\Delta s_k \approx \int_{\gamma AB} F(y)\,ds$$

can be interpreted mechanically as the potential energy of the system of rings and springs shown in Figure 1. Each of the N rings slides without friction on its own line; the neighboring rings are coupled by constant-tension springs whose tensions are given by the discretized values of $F_k = F(y_k)$. If P_N is minimal, each ring is in equilibrium, implying the balance of horizontal forces on the ring:

$$F(y_k)\sin\theta_k = F(y_{k+1})\sin\theta_{k+1},\quad k = 1,\ldots N;$$

in the continuous limit this gives

$$F(y)\sin\theta = \text{constant},$$

or, equivalently,

$$\frac{F(y)}{\sqrt{1+(y')^2}} = \text{constant}.$$

This idea (along with some others in a similar spirit) can be found in [1].

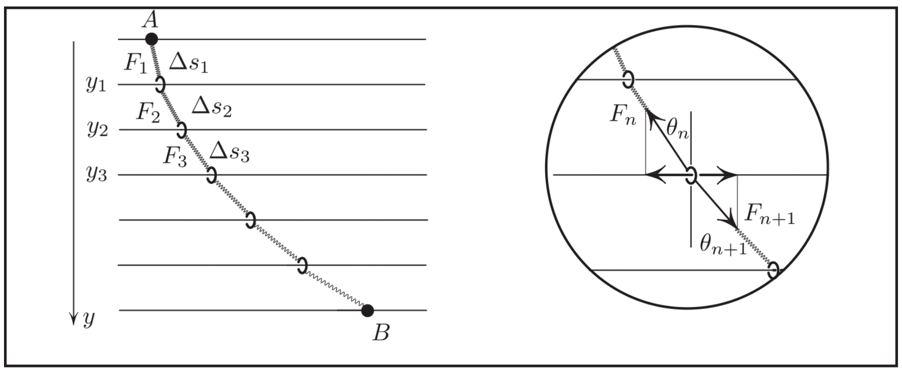

Figure 1. *Each spring has a prescribed tension F_k independent of its length Δs_k. The endpoints A and B are held fixed.*

References

[1] M. Levi, *Classical Mechanics with Calculus of Variations and Optimal Control*, AMS, Providence, Rhode Island, 2014.

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