Electrical Resistance and Conformal Maps

MATHEMATICAL CURIOSITIES
By Mark Levi

I would like to elaborate on the observation in my April 2023 article, titled "Conformal Deformation of Conductors."[1] Imagine a current-conducting sheet: negligibly thin, homogeneous, and isotropic. Let us cut a square out of the sheet and measure the resistance, as in Figure 1. The following fact is both fundamental and almost trivial:

$$\boxed{\begin{array}{c}\text{Squares of all sizes}\\ \text{have the same resistance.}\end{array}} \tag{1}$$

Indeed, dilation of the square changes the distance that the current must travel, and by the same factor as the width; these two effects cancel each other out — but *only* in $\mathbb{R}^2$. In $\mathbb{R}^3$, for example, dilating a cube by a factor λ divides the resistance (between the opposite faces) by λ, and in $\mathbb{R}$ the resistance multiplies by λ.

From now on, let the resistance of the square $=1$ ohm. Geometrically, resistance is a "measure of elongation": a rectangle whose resistance $=1$ must be a square (see Figure 2).

A classical theorem in complex analysis states that two annuli (see Figure 3) are conformally equivalent (i.e., they can be mapped onto one another by a conformal $1-1$ map) if and only if they have the same ratio of radii. A more general theorem states that two doubly connected "annuli"—like those in Figure 4—are conformally equivalent if they have the same modulus: a certain number that is associated with the region. I would like to point out that that the modulus is simply the electrical resistance.

[1] https://sinews.siam.org/Details-Page/conformal-deformation-of-conductors

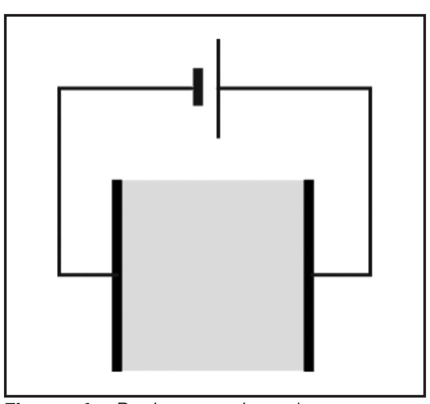

Figure 1. *Resistance—i.e., the necessary voltage to push through a unit of current—is measured between opposite sides (coated with a perfect conductor).*

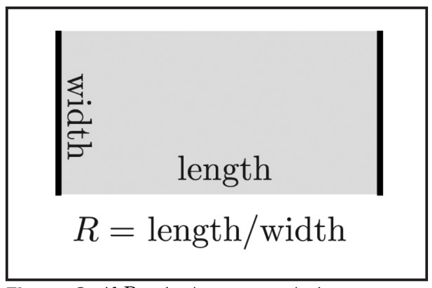

Figure 2. *If $R=1$, the rectangle is a square.*

To rephrase these theorems: *Two annular regions A and A' in Figure 4 are conformally equivalent: $A \sim A'$ if and only if they have the same electrical resistance between their inner and outer boundaries*:

$$R(A)=R(A'). \tag{2}$$

Idea of the Proof

In order to construct a conformal map $A \leftrightarrow A'$, let us push the current by applying voltages $V=0$ to the inner boundary and $V=1$ to the outer boundary.[2] For a large integer n, consider the equipotential lines h_i, $i=0,\ldots n$ that are spaced by the potential difference $1/n$ (see Figure 5); h_0 is the inner boundary and h_n is the outer boundary. Fix an arbitrary line v_0 of steepest descent of the electrostatic potential—the line of current—and let v_1 be the line of steepest descent that is chosen so that the current through the channel v_0v_1 is $1/n$. Continue adding current lines v_j, as in Figure 5, and stop at $j=m$ when the current through the channel v_mv_0 becomes $<1/n$. This last channel plays no role in the limit of $n\to\infty$.

[2] By doing so, we consider the solution of the Dirichlet problem in the annulus with prescribed boundary values 0 and 1.

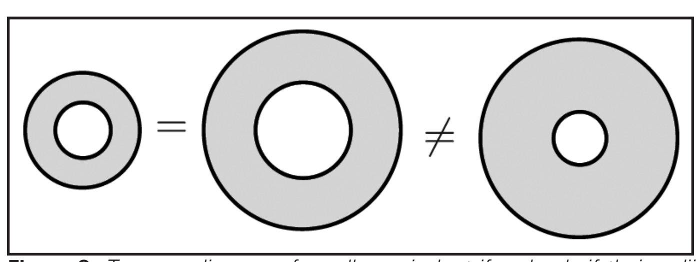

Figure 3. *Two annuli are conformally equivalent if and only if their radii have the same ratios.*

We divided the annulus into $n\times m$ infinitesimal curvilinear rectangles Q_{ij}, which we enumerate by the rectangle's layer i, $1\le i\le n$ and the channel j, $1\le j\le m$ (see Figure 5).

I claim that each curvilinear rectangle Q_{ij} *is a square in the limit of $n\to\infty$.* Indeed, the resistance is

$$R(Q_{ij})=\frac{\text{voltage drop}}{\text{current}}=\frac{1/n}{1/n}=1,$$

and a rectangle for which resistance $=1$ is a square (as indicated in Figure 2).

What is the resistance of A? Each channel has resistance n (being a stack of n squares), and with m channels in parallel,

$$R(A)=\frac{n}{m},$$

ignoring a small error due to the resistance of the last channel v_mv_0. The resistance thus has an almost combinatorial meaning.

To construct the map $A\leftrightarrow A'$, we divide A' into $n\times m'$ squares Q'_{ij}. If $R(A)=R(A')$, then $m=m'$; this allows a $1-1$ assignment of Q'_{ij} to Q_{ij}. The result is a discrete conformal map since it takes squares to squares.

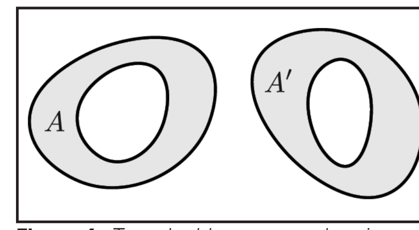

Figure 4. *Two doubly connected regions are conformally equivalent precisely when they have the same resistance between their inner and outer boundaries.*

Showing the converse: $A\sim A'$ implies $R(A)=R(A')$. We divide A into "squares" Q_{ij} as before, with $1\le i\le n$ and $1\le j\le m$. The conformal equivalence induces a division of A' into "squares" (by conformality) with the same $m'=m$ (since the map is $1-1$). Therefore, $n/m=n/m'$ and $R(A)=R(A')$. In short, (1) demonstrates that *the resistance is a conformal invariant*, as was already mentioned in the April 2023 article.

The figures in this article were provided by the author.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.

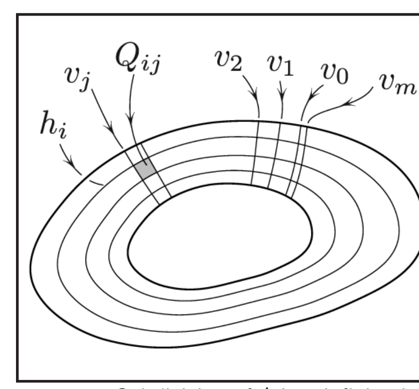

Figure 5. *Subdivision of A into infinitesimal squares Q_{ij}. Concentric "horizontal" lines h_i are equipotentials. Steepest descent "vertical" current lines v_j are added in a counterclockwise direction until the last line v_m. The "square" Q_{ij} is bounded by h_{i-1}, h_i and v_{j-1}, v_j.*