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Riemann Mapping Theorem by Steepest Descent

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This note presents a short proof of the following slightly weakened version of the Riemann mapping theorem.

Theorem 1 (Riemann). *Let D be the open region bounded by a simple closed analytic curve C in the complex plane, and let z_0 be a point in D. There exists an analytic function f that maps D onto the unit disk $\Delta = \{z : |z| < 1\}$ in a one-to-one fashion with $f(z_0) = 0$ and $f'(z_0) > 0$.*

This version of the theorem is weaker than the most general statement, which requires only the simple connectedness of D (see [**1**]). The map f in the theorem is unique, as the Schwarz lemma implies, but we are concerned here only with its existence.

Our proof relies on the existence of the solution to the Dirichlet problem in D (i.e., on the existence of a unique harmonic function that is continuous on the closure $\overline{D}$ of D and has prescribed values on the boundary ∂D of D). A different proof using the Dirichlet problem can be found in [**5**].

Construction of the map. Consider a harmonic function $u(z)$ vanishing on ∂D and possessing the following logarithmic singularity at z_0 (Figure 1; for simplicity of notation we choose $z_0 = 0$, which entails no loss of generality):

$$u(z) = \ln|z| + u_0(z). \tag{1}$$

Here u_0 is the solution of the Dirichlet problem in D for the boundary data $u_0(z) = -\ln|z|$, which is chosen to ensure that $u|_{\partial D} \equiv 0$. Physically, u can be interpreted as the stationary temperature of the heat-conducting homogeneous lamina D whose boundary is kept at zero temperature and whose interior contains a heat-sink of strength 2π placed at the origin. The function u is known as *Green's function* for the domain D.

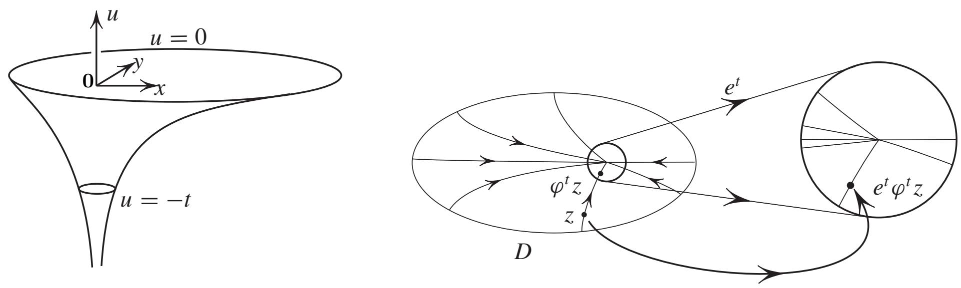

Figure 1. Construction of the Riemann mapping by steepest descent along Green's function.

With u so defined, consider the motion down the gradient of u with adjusted speed, corresponding to the following system of ODEs:

$$\dot{z} = -\frac{\nabla u}{|\nabla u|^2} \equiv F(z) \qquad (z \neq 0). \tag{2}$$

Here we identify the real vector $\nabla u = \langle u_x, u_y \rangle$ with the complex number $\nabla u = u_x + iu_y$. We will show that $\nabla u \neq 0$ for z in $D \setminus \{0\}$, so that the vector field F is well defined in the punctured domain. Furthermore, defining $F(0) = 0$ makes F continuous on D, since $|\nabla u| \to \infty$ as $z \to 0$.

Let φ^t be the time-t map associated with (2) (i.e., $\varphi^t z$ is the value at time t of the solution of (2) starting at z at $t = 0$). The Riemann map $f : D \mapsto \Delta_R = \{z : |z| < R\}$ is given, we claim, simply by

$$f(z) = \lim_{t\to\infty} e^t \varphi^t z, \qquad f(0) = 0, \tag{3}$$

where $R = e^{-u_0(0)}$.

Heuristic idea of the proof. First we observe that level curves of u remain such under the flow (2): $du/dt = \nabla u \cdot \dot{z} = -1$, and thus $\varphi^t D = \{z : u(z) < -t\}$. Since the leading term of u near 0 is $\ln |z|$, we conclude that $\varphi^t D$ is approximately a round disk:[1]

$$\{z : u_0(z) + \ln|z| < -t\} \approx \{z : u_0(0) + \ln|z| < -t\} = \{z : |z| < e^{-t}e^{-u_0(0)}\}.$$

Dilation by the factor e^t brings this to the disk Δ_R.

To explain the analyticity of f, we observe that the vector field $F(z)$ is analytic in D. Indeed, using the complex notation $\nabla u = u_x + iu_y$ introduced earlier we have

$$F(z) = -\frac{\nabla u}{\nabla u \overline{\nabla u}} = -\frac{1}{\overline{\nabla u}}.$$

Since u is harmonic in $D \setminus \{0\}$, $\overline{\nabla u}$ is analytic there (its real and the imaginary parts satisfy the Cauchy-Riemann equations). Because $\nabla u \neq 0$ in the punctured domain, we

[1] We do not define "approximate" here, since the subsequent rigorous proof does not rely on this heuristic discussion.

conclude that F is likewise analytic in $D \setminus \{0\}$. The remaining point $z = 0$ is a removable singularity of F, since F is continuous at $z = 0$. It follows that F is analytic in D. By the theorem on uniqueness and analytic dependence on initial data of solutions to ODEs with analytic right-hand sides, φ^t is one-to-one and analytic in z, hence so is $f_t = e^t \varphi^t$. This suggests that the limit $f = \lim_{t\to\infty} f_t$ is analytic as well. A rigorous proof of this fact is given in the next section. This completes our heuristic discussion.

Remark. This outline might create a (false, as it turns out) hope that the proof extends to higher dimensions, giving a diffeomorphism from any topological ball in $\mathbb{R}^n$ to a round ball. This hope is dashed by the fact that ∇u may vanish for topological balls in higher dimensional domains, as can be seen using ideas in [**4**]. In fact, our proof of $\nabla u \neq 0$ exploits the topology of the plane.

The proof. For a rigorous proof we have to verify (1) that ∇u does not vanish in $D \setminus \{0\}$, and (2) that $\lim_{t\to\infty} e^t \varphi^t z$ is an analytic one-to-one map from D to Δ_R.

(1) *The nonvanishing of* ∇u. It suffices to prove that ∇u does not vanish in an "annular" region

$$A_t = \{z : \ -t < u(z) < 0\} \tag{4}$$

for arbitrarily large t. According to the Poincaré index theorem [**2**],[2] the index of the vector field ∇u over the boundary of A_t equals the sum of the indices of rest points of the vector field ∇u inside A_t:

$$\mathrm{ind}_{\partial A_t} \nabla u = \sum_{a:\ \nabla u(a)=0} \mathrm{ind}_a \nabla u. \tag{5}$$

We show that the left-hand side of (5) vanishes, whereas $\mathrm{ind}_a \nabla u \leq -1$ for any critical point a of u, thus proving that the set of rest points of ∇u in A_t is empty. Starting with the left-hand side in (5), observe that the boundary ∂A_t consists of two level curves of u (namely, ∂D and the near-circle $\{z : \ u(z) = -t\}$) and that ∇u is normal to both curves. Moreover, $\nabla u \neq 0$ on both curves, as we prove in the next paragraph. Now the index of a normal vector field along a closed curve oriented in the counterclockwise direction is -1. Since the two boundary curves of A_t are oriented in opposite directions (as one travels along the boundary curve, the domain has to be on one's left), we conclude that the left-hand side in (5) is $1 - 1 = 0$. On the other hand, $\mathrm{ind}_a \nabla u \leq -1$ at any zero a of ∇u, according to Theorem 2 at the end of this note.

It remains to show that $\nabla u \neq 0$ on ∂A_t. For the inner boundary, where $u = -t$, this is immediate from (1) for sufficiently large t. Because the outer boundary ∂D is analytic, u can be extended to a harmonic function in a neighborhood of any boundary point p of D (see, for example, [**3**]). It follows that $u = \mathrm{Re}\ g(z)$ for some complex function g analytic in some open disk centered at p. Now, if $\nabla u(p) = 0$, then $g'(p) = 0$, implying that $g(z) = c_n(z-p)^n + \cdots$ with $n \geq 2$. Thus for z near p

$$u(z) = \mathrm{Re}\ c_n(z-p)^n + \cdots. \tag{6}$$

This function takes alternating signs in sectors, as illustrated in Figure 2. Since ∂D is a smooth curve passing through p and since there are $2n$ (≥ 4) sectors, we conclude that $u(z) > 0$ at some points inside D (Figure 2). But $u \leq 0$ on ∂A_t and by the maximum principle we must have $u < 0$ in the interior of A_t, a contradiction. The proof of $\nabla u \neq 0$ is complete.

[2] See the last section of this note for the definition of the index and for the statement of Poincaré's theorem.

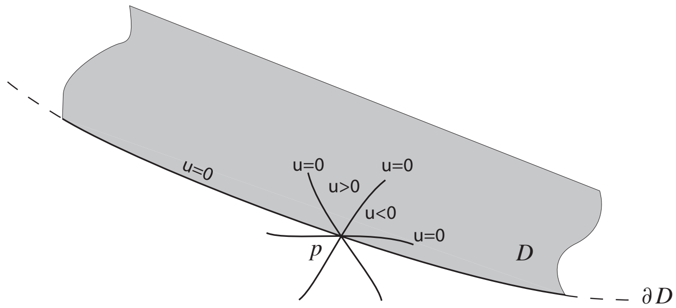

Figure 2. Presence of a critical point on ∂D forces u to vanish at some points inside D.

(2) *Proof that* $f(z) = \lim_{t\to\infty} e^t\varphi^t z$ *is a one-to-one and analytic map from D to* Δ_R. We could prove this directly by showing that $f_t(z) = e^t\varphi^t z$ "slows down" fast enough as $t \to \infty$ for each fixed z, so that the properties of f_t survive in the limit. We choose the quicker approach of compactifying the time interval. We compress the "eternity" $0 \le t < \infty$ into one second: $0 \le \tau < 1$ by the rescaling $\tau = 1 - e^{-t}$; thus $t = \infty$ corresponds to $\tau = 1$. We show that the flow extends continuously to the *closed* time-interval [0, 1] (this is the key), and then appeal to the theorem on analytic dependence on initial conditions, concluding that f is well defined and analytic. We now proceed with the details.

The dilated variable $Z(t, z) := e^t\varphi^t z$ satisfies the ODE

$$\dot{Z} = Z + e^t F(e^{-t} Z), \tag{7}$$

as one checks by substituting $\varphi^t z = e^{-t} Z$ into (2). As remarked earlier, F is analytic in D. From (1) we conclude that the leading Taylor coefficient of F at 0 is -1: $F(z) = -z + Q(z)$, where $Q(z) = q_2 z^2 + \cdots$ for z near 0. Substituting this expression for F into (7) we obtain

$$\dot{Z} = e^t Q(e^{-t} Z) \tag{8}$$

(i.e., the linear terms in (7) cancel). This cancellation was to be expected, because the exponential rate of contraction by φ^t approaches the exponential rate of expansion by e^t for large t, reflecting the "slowing down"of $f_t(z)$ mentioned earlier. In terms of the new time $\tau = 1 - e^{-t}$, (8) translates to

$$W' = (1-\tau)^{-2} Q((1-\tau)W), \tag{9}$$

where $W(\tau) = Z(t(\tau))$ and $' = d/d\,\tau$ (we used $d/d\,t = (1-\tau)d/d\,\tau$).

Since Q starts with quadratic terms, the right-hand side in (9) is continuous for τ in [0, 1], including at $\tau = 1$. The time-one map $f := W_{\tau=0} \mapsto W_{\tau=1}$ is thus well defined. This map is one-to-one and analytic, since the right-hand side of (9) is analytic in W. Finally, $f(D) = \{W : |W| < R\}$, with $R = e^{-u_0(0)}$. In fact, $\varphi^t D = \{z : u(z) < -t\}$, so the dilated domain, consisting of $Z = e^t z$ for z in $\varphi^t D$, is given by

$$e^t\varphi^t D = \{Z : u_0(e^{-t} Z) + \ln|e^{-t} Z| < -t\}. \tag{10}$$

Expressing (10) in terms of τ and simplifying, we obtain

$$e^t\varphi^t D = \{W : e^{u_0((1-\tau)W)}|W| < 1\}.$$

For $\tau = 1$—an allowed value!—this set is precisely the disk $|W| < e^{-u_0(0)} = R$, as claimed. ■

Index of a critical point of a harmonic function. The following fact was used in the foregoing discussion:

Theorem 2. *The Poincaré index of any zero of the gradient vector field of a harmonic function u in $\mathbb{R}^2$ is at most -1.*

Proof. Let $\nabla u(0) = 0$.[3] Since u is harmonic, the complex function $\overline{\nabla u}$ is analytic near 0, whence it is given by a Taylor series near 0:

$$\overline{\nabla u}(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots ,$$

where $k \geq 1$. Thus $\mathrm{ind}_{z=0}\overline{\nabla u} = k$. Since conjugation changes the sign of the Poincaré index, as follows from the definition of this concept [**2**] and from the fact that conjugation changes the sign of the argument of a complex number, we have

$$\mathrm{ind}_{z=0}\nabla u = -k \leq -1. \tag{11}$$

■

Poincaré's index theorem. We give some minimal background on the index of a vector field in the plane. Details and proofs can be found in [**2**].

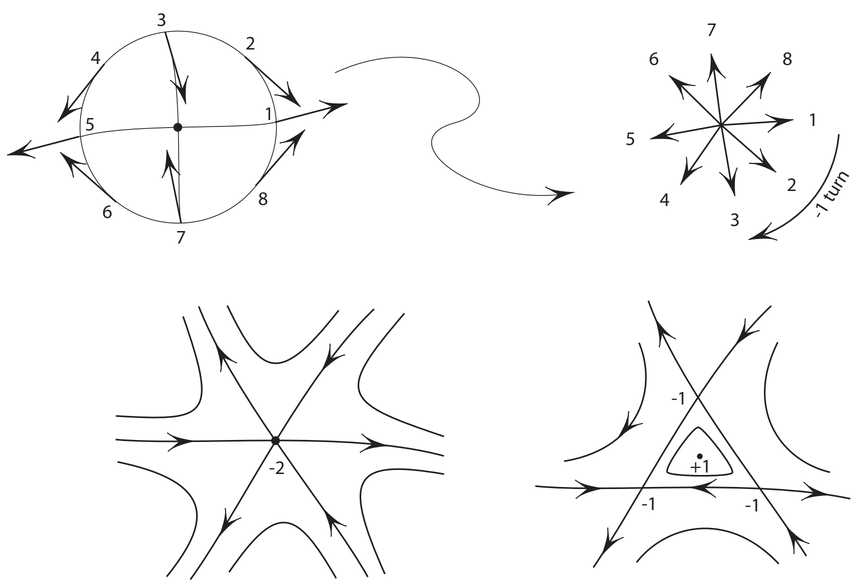

Figure 3. Poincaré index of the saddle is -1; for the degenerate saddle with $k = 2$ in (11) the index is -2. Under perturbation this degenerate saddle splits into three saddles and a center; the total index remains unchanged.

Let v be a continuous vector field on a domain D in $\mathbb{R}^2$, and let γ be a simple closed oriented curve in D such that v does not vanish on γ. *The index* $\mathrm{ind}_\gamma v$ *of* v *on* γ is, by definition, the integer number of revolutions made by the vector $v(z)$ as the point z traverses γ exactly once in the prescribed direction (Figure 3). Consider an isolated rest point a of the vector field v, and let γ_a be a simple closed curve surrounding a but having no other rest points in its interior. This time let γ_a have the counterclockwise

[3] We assume, without loss of generality, that the rest point is at the origin.

orientation. *The index* $\mathrm{ind}_a v$ *of the vector field* v *at the rest point* a is given by

$$\mathrm{ind}_a v = \mathrm{ind}_{\gamma_a} v.$$

The index does not depend on the choice of the curve γ_a (see [**2**]).

To formulate Poincaré's theorem, consider a domain D in $\mathbb{R}^2$ whose boundary is the disjoint union of finitely many smooth simple closed curves (a smoothly bounded annulus is an example). We give each curve an orientation such that the domain remains on one's left when the curve is traversed with that orientation. More formally, a tangent vector whose direction agrees with the prescribed orientation and an inward normal vector form a right-handed frame. For the example of the annulus the outer boundary's orientation is counterclockwise, while the inner boundary's orientation is clockwise. With this convention, *the index around* ∂D *of a vector field defined in a neighborhood of* $\overline{D}$ *is the sum of its indices on the component curves of the boundary.*

Theorem 3 (Poincaré Index Theorem). *Let* v *be a continuous vector field defined on a neighborhood of the closure of a domain* D *in* $\mathbb{R}^2$ *that is bounded by finitely many smooth simple closed curves. Assume that all rest points of* v *are isolated and that none lie on* ∂D*. Then the Poincaré index of* v *around* ∂D *equals the sum of indices of all rest points inside* D.

A proof of the theorem can be found in [**2**].

ACKNOWLEDGMENT. This research was supported by the NSF grant DMS-9704554.

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Entire Functions that Tend to Zero on Every Line

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1. INTRODUCTION. Although every bounded entire (holomorphic) function on $\mathbb{C}$ is constant (Liouville's theorem), it has been known for more than a hundred years that there exist nonconstant entire functions f such that $f(z) \to 0$ as $z \to \infty$ along every line through 0 (see, for example, Lindelöf's book [**10**, pp. 119–122] of 1905).