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ROTATIONAL ELASTIC DYNAMICS*

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The combined dynamical effects of elasticity and a rotating reference frame are explored for structures in a zero gravity environment. A simple yet general approach to modeling is presented, and this approach is applied to analyze in detail the dynamics of a specific prototypical structure. Energy dissipation is included and its effects are studied in detail in a model problem. Bifurcations and stability are analyzed as well.

1. Introduction

There is now a fairly general awareness among aerospace engineers that the dynamics of the complex spacecraft currently in production and on the drawing boards will be greatly influenced by continuum mechanical effects such as elasticity. Indeed as the designs being contemplated increase in size and complexity [13] the dynamic effects of flexible members become more important, and recognizing this, many researchers over the last decade have focused their efforts on obtaining new methods for the design, analysis, and control of flexible mechanisms. Space does not permit (nor would it be in keeping with the main purpose of this paper) to survey the vast literature on flexible space structures; the interested reader can get some idea of research activity in this area by referring to any one of a number of collections of papers and conference proceedings, such as [12].

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The purpose of this paper is to describe recent research which has been aimed at developing a mathematical theory of the rotational dynamics of complex mechanical systems which include articulated and elastic components. Our objective in this research has been to carry out a study of the global qualitative dynamics of such systems in sufficient depth as to allow predictions regarding the stability and asymptotic behavior of spacecraft due to a variety of energy dissipation mechanisms such as viscoelastic material damping of vibrations of elastic parts. We believe that historical evidence points to the value of developing a fairly complete global asymptotic stability theory of this type since there are numerous examples of missions in space which did not achieve their stated objectives because certain long term mechanical effects were never adequately taken into account in the mission planning. Explorer I, the first successfully launched American satellite, provides the best known example of such untoward behavior. Upon achieving earth orbit, the pencil-shaped satellite was supposed to rotate about its major

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axis of inertia. Before it had completed one orbital revolution, however, radio signals indicated that a tumbling motion had developed and was increasing in amplitude.

While explanations of Explorer's errant behavior have been offered by a number of researchers (see e.g. references to this problem in [7]), we are aware of no attempts to obtain a rigorous mathematical analysis of such occurrences, with the exception of [3], where some of the results of the present paper were announced. We also mention a recent paper [8] which deals with a somewhat different, although related aspect of a similar problem.

The present paper is organized as follows. In section 2 we derive equations describing the rotational dynamics of complex structures. These include the general effects of inertial forces created by rotation of the reference frame. Section 3 focuses on a general theoretical framework for Lagrangian mechanics with damping. In section 4, a simple structure consisting of a rigid body with an elastic beam appendage is studied, and we present what we believe is the simplest reasonable continuum mechanical model of such a system undergoing three degree of freedom rotations. The asymptotic steady state dynamics for this system are studied in section 5, and in section 6 we present a detailed analysis of the qualitative dynamics of a closely related model having only one rotational degree of freedom.

2. The rotational dynamics of complex structures

In this section we will derive equations of motion for a class of structures consisting of elastic, fluid or rigid components. While these equations are completely general, they are most useful in describing any structure whose configuration is conveniently specified by the position of the structure with respect to a moving coordinate frame together with the position of that frame in space, i.e. relative to some fixed inertial system. We fix an orthonormal basis forming an inertial frame

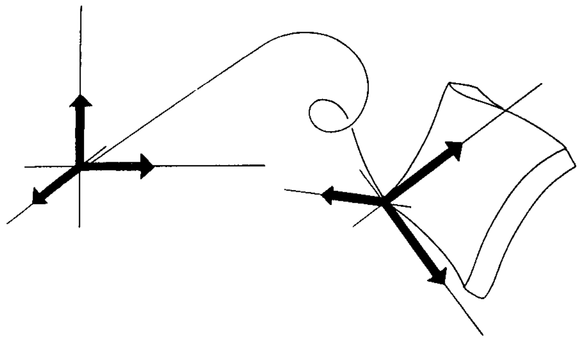

Fig. 1. The body frame is translated and rotated with respect to the inertial frame.

("space frame"), and choose a "body" coordinate system designated by a set of orthonormal vectors g_1, g_2, g_3. Choice of the g_i's will depend on the particular problem at hand, and it affects the simplicity of resulting equations. In section 4, where we treat an example of a rigid body with an elastic appendage, we affix the body frame g_i to the rigid body, although other choices are possible [6].

The position and orientation of the body frame (with respect to the chosen inertial frame) may be described at each time t by an element of the special Euclidean group, SE(3, $\mathbb{R}$), represented by a 4×4 matrix

$$X(t) = \begin{pmatrix} Y(t) & y(t) \\ 0 & 1 \end{pmatrix}$$

(see fig. 1), where $Y(t) \in SO(3)$ is an orthogonal matrix describing the orientation of the body frame and $y \in \mathbb{R}^3$ is the position of the origin of that frame in space (cf. [5]).

Therefore, if a point P is given by the vector u in the body frame and by vector U in the space frame, then u and U are related via

$$\hat{U} = X\hat{u},$$

where $\hat{U} = \binom{U}{1}$ and $\hat{u} = \binom{u}{1}$.

The positions of various elements of our flexible structure relative to the body frame will be described by a vector function $u(z, t) =$

$u(z_1, z_2, z_3, t)$ denoting the position at time t of each particle whose "unperturbed" position is at $z = (z_1, z_2, z_3)$. Here "unperturbed" can mean either initial or undeformed, depending on one's choice. In section 4, z will denote the neutral position of a particle of the flexible structure.

In summary, we consider systems whose configuration space is given by $\{q\} = \{(Y, u, y)\} = \mathrm{SE}(3) \times \mathrm{C} \equiv \mathrm{SO}(3) \times \mathbb{R}^3 \times C$, where $C = \{u(\cdot, t)\}$ is a suitably defined function space whose elements are functions $u(z)$ describing the configuration of the body relative to the body frame.

We describe now the kinematics of the system and give an expression for the kinetic energy. The evolution of the matrix $X(t) \in \mathrm{SE}(3)$ can be described by a differential equation

$$\dot{X}(t) = \begin{pmatrix} Y(t)\Omega(t) & \dot{y}(t) \\ 0 & 0 \end{pmatrix},$$

where $\Omega(t)$ is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

of angular velocities ω_i about the corresponding body axes g_i, $i = 1, 2, 3$.

The inertial coordinates $U(z, t)$ of a point P are related to its body coordinates $u(z, t)$ via

$$\hat{U}(z, t) = X\hat{u}(z, t)$$

and the corresponding velocities are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U} = \dot{X}\hat{u} + X\hat{u}_t = \begin{pmatrix} Y(\Omega u + u_t) + \dot{y} \\ 0 \end{pmatrix},$$

where u_t denotes the partial derivative with respect to t.

The kinetic energy of the body is then given by

$$T(q, \dot{q}) = \tfrac{1}{2}\int_B \|\dot{U}\|^2 \,\mathrm{d}m$$
$$= \tfrac{1}{2}\int_B \| Y(\Omega u + u_t) + \dot{y} \|^2 \,\mathrm{d}m, \qquad (2.1)$$

where B denotes the point set comprising the body (at time t) described in the body coordinate system, and $\mathrm{d}m$ is the mass distribution in this coordinate system.

The kinetic energy of almost any rotating structure will be of this form, as it does not depend on the constitutive relations governing the structure itself. More refined dynamical models, as treated in subsequent sections, will embody structural information in the expression for *potential* energy

$$V(q) = V(Y, u, y)$$

(which we assume to be independent of $\dot{q}$). Without specifying the form of V at this point, we prove the main theorem of this section.

Theorem 2.1. Let A be defined by

$$A(t) = \int_B \left((\Omega u + u_t)u^{\mathrm{T}}\right)_{\mathrm{a}} \mathrm{d}m,$$

where $M_{\mathrm{a}} = \frac{1}{2}(M - M^{\mathrm{T}})$ denotes the anti-symmetric part of a matrix M. Equations of motion of any system whose configuration space is $\{q\} = \{(Y, u, y)\} = \mathrm{SO}(3) \times C \times \mathbb{R}^3$ as above are given by

$$\dot{A}(t) + [\Omega(t), A(t)] - \int_B \left(Y^{\mathrm{T}}\ddot{y}u^{\mathrm{T}}\right)_{\mathrm{a}} \mathrm{d}m$$
$$= \mathscr{T} - \left(Y^{-1}V_Y\right)_{\mathrm{a}}, \qquad (2.2)$$

$$u_{tt} + \Omega^2 u + \dot{\Omega}u + 2\Omega u_t + Y^{\mathrm{T}}\ddot{y} = F - \frac{\delta V}{\delta u}, \qquad (2.3)$$

$$\int_B Y\left(\Omega^2 u + \dot{\Omega}u + 2\Omega u_t + u_{tt}\right) + \ddot{y}\,\mathrm{d}m = f - \frac{\partial V}{\partial y}, \qquad (2.4)$$

$$\dot{Y} = Y\Omega, \qquad (2.4)'$$

where V_Y is the matrix of partial derivatives $\partial V / \partial Y_{ij}$, and $\delta V / \delta u$ is the Frechet derivative of V with respect to u. In the skew-symmetric matrix

$$\mathscr{T} = \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0, \end{pmatrix}$$

τ_i is the net nonconservative torque applied to the system about the body axis g_i, $F = F(z)$ is the distributed nonconservative force density acting on the particle positioned at $u(z, t)$ expressed in the body coordinate system, and f is the net nonconservative exogenous force.

Remark. In subsequent sections nonconservative forces will arise due to viscoelastic damping.

Before giving the proof, we rewrite these equations so as to provide a clearer picture of the physical situation. Let $\omega = (\omega_1, \omega_2, \omega_3)^T$ be the angular velocity of the body frame *expressed in that frame*; ω is related to the angular velocity matrix $\Omega = Y^{-1}\dot{Y}$ as follows:

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Denote by S the operator taking a skew-symmetric matrix Ω into vector ω: $S\Omega = \omega$. One easily verifies the following:

Lemma 2.1. Given any pair A, B of skew-symmetric 3×3 matrices and any pair u, v of 3-vectors, the following identities hold:

(i) $S([A, B]) = S(A) \times S(B)$;
(ii) $S(uv^T - vu^T) = v \times u$;
(iii) $Au = S(A) \times u$.

Applying the operator S to eq. (2.2) and using lemma 2.1 on eqs. (2.2)–(2.4) we obtain

Corollary 2.1. Equations of motion (2.2)–(2.4) are equivalent to

$$\dot{a}(t) + \omega(t) \times a(t) + \int_B u(z, t) \times Y^{-1}\ddot{y}(t)\,\mathrm{d}m = S\left(\mathscr{T} - \left(Y^{-1}V_Y\right)_a\right), \qquad (2.5)$$

$$u_{tt} + \omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t + Y^{-1}\ddot{y} = F - \frac{\delta V}{\delta u}, \qquad (2.6)$$

$$\int_B [Y(\omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t + u_{tt}) + \ddot{y}]\,\mathrm{d}m = f - \frac{\partial V}{\partial y}, \qquad (2.7)$$

where a is defined by

$$a(t) = \int_B u \times (u_t + \omega \times u)\,\mathrm{d}m.$$

Written in this way eqs. (2.5)–(2.7) give an explicit description of the inertial forces on the mechanical system viewed in the moving body frame. Introducing the derivation $D = \mathrm{d}/\mathrm{d}t(\cdot) + \omega \times (\cdot)$, we obtain yet another rendering of these equations.

Corollary 2.2. Equations of motion (2.2)–(2.4) are equivalent to

$$\int_B u \times (D^2 u + Y^{-1}\ddot{y})\,\mathrm{d}m = S\left(\mathscr{T} - \left(Y^{-1}V_Y\right)_a\right), \qquad (2.5)'$$

$$D^2 u + Y^{-1}\ddot{y} = F - \frac{\delta V}{\delta u}, \qquad (2.6)'$$

$$\int_B (YD^2 u + \ddot{y})\,\mathrm{d}m = f - \frac{\partial V}{\partial y}. \qquad (2.7)'$$

The motions of any complex structure undergoing free or forced rotation are described by eqs. (2.5)–(2.7). This formulation is thus fairly general, and it can incorporate external forces and torques (due, for example, to gravitational and magnetic fields) and internal forces (due, say, to actuation of joints or the constitutive properties of the material). In the next section 3 we will incorporate dissipative effects in this formulation, and in section 4, we shall develop a complete dynamical model of a system where the constitutive relations are those of a simple damped beam.

Proof of theorem 2.1. The proof will be given in two parts: First, the treatment of the inclusion $Y \in \mathrm{SO}(3)$ as the holonomic *constraint* onto SO(3) as a submanifold of GL(3), with an appropriate modification of the Lagrange equations (lemma 2.2), and second, the utilization of the *left-invariance* of (the rigid path of) kinetic energy to simplify the resulting equations (lemma 2.4).

We would like to write the equations of motion of the structure in the Lagrangian form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q}, \quad q = (Y, u, y).$$

An appropriate modification of the Lagrange equations is needed, however, to account for the fact that Y is constrained to the submanifold $\mathrm{SO}(3) \subset \mathrm{Gl}(3)$. Such a modification is unnecessary if the Lagrangian equations are expressed in terms of a local coordinate system on the constraint manifold—cf. [1], page 77. This approach ignores the symmetry in our system of equations, however. The following Lemmas 2.2 and 2.4 are key to our proof of theorem 2.1:

Lemma 2.2. Any extremal $q(t)$ of the action $\int L(q, \dot{q})\,\mathrm{d}t$ with q constrained to a submanifold M_0 of a Riemannian manifold M satisfies the differential equation

$$\pi_q\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) = 0,$$

where π_q is the normal projection from $T_q M$ onto $\mathrm{T}_q M_0$.

We omit a straightforward proof. Corresponding to the three components of $q = (Y, u, y)$ we obtain three components for the equations of motion:

$$P_Y\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{Y}} - \frac{\partial T}{\partial Y} + \frac{\partial V}{\partial Y}\right) = 0, \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta T}{\delta \dot{u}} - \frac{\delta T}{\delta u} = -\frac{\delta V}{\delta u}, \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = -\frac{\partial V}{\partial y}, \tag{2.10}$$

where P_Y is the orthogonal projection from $\mathrm{T}_Y\mathrm{Gl}(3)$ onto $\mathrm{T}_Y\mathrm{SO}(3)$ in the trace norm $\langle A, B\rangle = \mathrm{Tr}\, A^{\mathrm{T}}B$. Here $\partial T/\partial \dot{Y}$, $\partial T/\partial Y$ are the derivatives with respect to the standard Riemannian structure on TGL(3); they can be represented as matrices of partial derivatives: $\partial T/\partial \dot{Y} = (\partial T/\partial \dot{Y}_{ij})$, and $\partial T/\partial Y = (\partial T/\partial Y_{ij})$, $\delta T/\delta u$ denotes the Frechet derivative of T with respect to the distributed parameter u. The expression for the projection P_Y is provided by

Lemma 2.3. For any $Y \in \mathrm{SO}(3)$ and $A \subset T_Y\mathrm{Gl}(3)$

$$P_Y A = Y(Y^{-1}A)_{\mathrm{a}}, \tag{2.11}$$

where $X_{\mathrm{a}} = \frac{1}{2}(X - X^{\mathrm{T}})$ is the anti-symmetric part of X.

Proof. This follows from decomposing the Lie algebra of gl(3) as the orthogonal direct sum of symmetric and skew symmetric matrices. ($\mathrm{gl}(3) = \mathrm{so}(3) \oplus \mathrm{so}(3)^{\perp}$.) ■

Lemma 2.4. Suppose T is a left-invariant function on $\mathrm{TGl}(n)$, i.e. suppose there is a function K defined on $\mathrm{gl}(n)$ (= the Lie algebra of $\mathrm{Gl}(n)$ = space of real $n \times n$ matrices) such that $T(Y, \dot{Y}) = K(\Omega)$ where $\Omega = Y^{-1}\dot{Y}$. Then for $(Y, \dot{Y}) \in \mathrm{TSO}(n)$

$$P_Y\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{Y}} - \frac{\partial T}{\partial Y}\right) = Y\left(\frac{\mathrm{d}}{\mathrm{d}t}M + [\Omega, M]\right),$$

where $M = (\partial K/\partial \Omega)_{\mathrm{a}}$, and $\partial K/\partial \Omega$ is the derivative of K with respect to Ω evaluated at $\Omega = Y^{-1}\dot{Y}$.

Remark. There is an orthogonal direct sum decomposition: $\mathrm{gl}(n) = \mathrm{so}(n) \oplus \mathrm{so}(n)^{\perp}$, where $\mathrm{so}(n)$ = the Lie algebra of $n \times n$ skew-symmetric matrices, and where orthogonality is defined in terms of the trace form inner product $\langle A, B\rangle = \mathrm{Tr}\, AB^{\mathrm{T}}$ defined on $\mathrm{gl}(n)$. If the function K appearing in the statement of lemma 2.4 can be decomposed as $K = K_1 + K_2$ where K_i depends only on the ith component ($i = 1, 2$) in this orthogonal direct sum, then $M = K_1'(\Omega)$, where by K_1' we mean the derivative of K_1 with respect to the natural differentiable structure on $\mathrm{so}(n)$ defined in terms of the Killing form.

Proof. $\mathrm{d}(\partial T/\partial \dot{Y})/\mathrm{d}t$ and $\partial T/\partial Y$ may be thought of as elements in the cotangent bundle

$T^*\mathrm{Gl}(n)$. Making the usual identifications, the standard Riemannian structure on $\mathrm{Gl}(n)$ may be prescribed explicitly in terms of the trace form, and we may write $\mathrm{d}(\partial T/\partial \dot{Y})/\mathrm{d}t$, $\partial T/\partial Y$ and $\partial K/\partial \Omega$ all as $n \times n$ matrices

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{Y}} = (Y^{-1})^{\mathrm{T}}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \Omega}\right) - \Omega^{\mathrm{T}}\frac{\partial K}{\partial \Omega}\right],$$

$$\frac{\partial T}{\partial Y} = -(Y^{-1})^{\mathrm{T}}\frac{\partial K}{\partial \Omega}\Omega^{\mathrm{T}}.$$

For $(Y, \dot{Y}) \in \mathrm{TSO}(n)$, we have $(Y^{-1})^{\mathrm{T}} = Y$ and $\Omega^{\mathrm{T}} = -\Omega$, and the result follows from lemma 2.3. ■

The proof of theorem 2.1 proceeds as follows. Using left-invariance of the first term in the expression for kinetic energy (cf. (2.1))

$$T = \tfrac{1}{2}\int_B \|\Omega u + u_t\|^2 + 2\left(\Omega u + u_t, Y^{-1}\dot{y}\right) + \|\dot{y}\|^2\,\mathrm{d}m,$$

we obtain eq. (2.2) as a consequence of lemma 2.4. The remaining equations (2.3) and (2.4) follow by direct computation from eq. (2.9) and (2.10). We omit the details.

3. Lagrangian mechanics with damping

A major advantage of Lagrangian versus Newtonian mechanics is the invariance of Lagrange's equations with respect to coordinate changes; it is this invariance that facilitates significantly the derivation of our equations of a motion. It is thus desirable to have the extension to the dissipative case. Such a modification is described in [9, 11]; we reproduce it here in a slightly more general form.

Let $D(q, \dot{q})$ be the so-called dissipation function defined as follows:

$\dot{q}D_{\dot{q}}$ = rate of dissipation of energy per second.

(One can think of $D_{\dot{q}}$ as the generalized dissipation force, and $\dot{q}$ = velocity. The above simply says: *velocity · force = power*.)

Let $L(q, \dot{q})$ be the Lagrangian of the system. The equations governing the system are

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{\dot{q}} - L_q + D_{\dot{q}} = 0. \tag{3.1}$$

Remark 3.1. If D is quadratic in $\dot{q}$ (as will be the case in the application presented in the next section), then

$$D = \tfrac{1}{2}\dot{q}D_{\dot{q}} = \tfrac{1}{2}(\text{rate of dissipation}).$$

Eqs. (3.1) are consistent with the definition of D, as shown by

Theorem 3.1. If $E(q, \dot{q})$ is the total energy of the system, then

$$\frac{\mathrm{d}}{\mathrm{d}t}E(q, \dot{q}) = -\dot{q}D_{\dot{q}}.$$

Proof. E is given by the Legendre transform* in $\dot{q}$ of L:

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L.$$

Differentiation by time gives

$$\dot{E} = \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - L_q\dot{q} - L_{\dot{q}}\ddot{q} = -\dot{q}D_{\dot{q}};$$

we used eq. (3.1) in the last step. ■

Eq. (3.1) has the same invariance property as the conservative Lagrange equations.

Theorem 3.2 [11]. The dissipative Lagrangian system (3.1) is invariant under the change of variables $q = q(Q)$. More precisely, if $q(t)$ satisfies (3.1), then $Q(t)$ satisfies equations of the same form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_{\dot{Q}} - \mathscr{L}_Q + \mathscr{D}_{\dot{Q}} = 0, \tag{3.1$'$}$$

* Usually E is expressed as a function of q, $p = \partial L/\partial \dot{q}$; we keep $\dot{q}$ rather than p here.

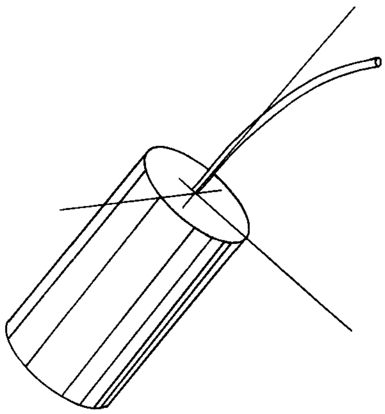

Fig. 2. A rigid body with cantilevered beam attachment.

where $\mathscr{L}(Q, \dot{Q}) = L(q(Q), q'(Q)\dot{Q})$ and $\mathscr{D}(Q, \dot{Q}) = D(q(Q), q'(Q)\dot{Q})$.

Proof. A simple calculation shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_{\dot{Q}} - \mathscr{L}_Q + \mathscr{D}_{\dot{Q}} = q'(Q)^{\mathrm{T}}\left(\frac{\mathrm{d}}{\mathrm{d}t}L_{\dot{q}} - L_q + D_{\dot{q}}\right).$$

■

4. A rotating rigid body with a beam attachment

Consider the spacecraft depicted in fig. 2. The key features of this structure are a rigid body to which a flexible cantilevered beam-like appendage of length l is attached.

As we have mentioned in section 2, we affix the moving frame to the rigid body. More specifically, we place the origin of the frame at the point of attachment of the beam and align the $z_3 \equiv z$-axis along the undeflected beam. Viewing this cantilevered beam as essentially a one-dimensional object, we describe the elastic deformations $u(z_3, t) \equiv u(z, t)$ with respect to the coordinate axes $(z_1, z_2, z_3) \equiv (x, y, z)$ depicted in fig. 2. More precisely, $u(z, t)$ is the position of the particle whose neutral position is at $(0, 0, z)$. The decomposition of the system into the rigid part and the elastic beam corresponds to the decomposition of kinetic energy (2.1) into the sum of rotational and translational energies of the rigid part and the energy of the beam. (Note that $u(\cdot, t)$ restricted to the rigid component is just the identity mapping for all t.) We have

$$T = \tfrac{1}{2}\omega^{\mathrm{T}} I \omega + \mathsf{m}_{\mathrm{b}} \dot{y}^{\mathrm{T}} Y \Omega \bar{c} + \tfrac{1}{2}\mathsf{m}_{\mathrm{b}} \|\dot{y}\|^2 + \tfrac{1}{2}\int_0^l \|Y(\Omega u + u_t) + \dot{y}\|^2 \,\mathrm{d}z,$$

where $S(\Omega) = \omega$, $\bar{c}$ is the center of mass of the rigid body in the body frame, m_{b} is the mass of the rigid body, we have scaled the linear mass density of the beam to be one, and the inertia tensor with respect to the body frame is given by

$$I = \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{pmatrix},$$

where I_x is the moment of inertia with respect to the x-axis, etc. (cf. [1]).

Since the beam is clamped at the origin of the (x, y, z)-coordinate system and free at its other end, the following boundary conditions are assumed:

$$u_i(0, t) = \frac{\partial u_i}{\partial z}(0, t) = \frac{\partial^2 u_i}{\partial z^2}(l, t) = \frac{\partial^3 u_i}{\partial z^3}(l, t) = 0, \quad i = 1, 2, \tag{4.1}$$

and $u_3(0, t) = \partial u_3(l, t)/\partial z = 0$. These boundary conditions are standard in the theory of clamped-free beams. Here u_1, u_2, u_3 are the deflections: $u = (u_1, u_2, z + u_3)$. Note that u_3 is *not* the z-coordinate of u.

The equations of motion for our rotating satellite are obtained by incorporating (2.5)–(2.7) into the formalism of Lagrangian mechanics, as discussed in the previous section. Thus we look for extremals of the Lagrangian $L = T - V$ with kinetic energy T given above and potential V given by the *strain energy*

$$V(u) = \tfrac{1}{2}\int_0^l \left[\mu_1 (u_1'')^2 + \mu_2 (u_2'')^2 + \mu_3 (u_3')^2\right] \mathrm{d}z, \quad ' = \frac{\partial}{\partial z}, \tag{4.2}$$

where only quadratic terms were retained and the material is assumed to obey a linear Hooke's law. Here μ_1 (respectively μ_2) gives the bending elasticity within the xz-plane (yz-plane respectively), and μ_3 is the Hooke's constant giving the beam's stretching elasticity. Unless the beam is abnormally thick, $\mu_3 \gg \mu_1, \mu_2$. The dissipation function is given by

$$D = D(u, \dot{u}) = \tfrac{1}{2}\int_0^l k_1(\dot{u}_1'')^2 + k_2(\dot{u}_2'')^2 + k_3(\dot{u}_3')^2 \, \mathrm{d}z, \qquad \cdot = \frac{\partial}{\partial t}, \quad ' = \frac{\partial}{\partial z}, \tag{4.3}$$

where $\dot{u}_1''$, $\dot{u}_2''$ can be thought of as the rates of change of appropriate curvatures, while $\dot{u}_3'$ is the rate of change of the contraction coefficient u_3'. The k_i's are positive constants reflecting the rates of energy dissipation due to deformation of material in the beam.

Remark 4.1. It might seem at a first glance that the first two terms in the integral (4.2) should be replaced by a quadratic form (in u_1'', u_2''),

$$a(u_1'')^2 + 2bu_1''u_2'' + c(u_2'')^2;$$

however, by properly turning the body coordinate system around the z-axis, we can diagonalize this form. In fig. 3, the x-axis is chosen along the direction in which the beam bends most easily ($\mu_1 \le \mu_2$); consequently, the beam offers the stiffest resistance to bending within the yz-plane.

Remark 4.2. We must point out that the beam was assumed to be of uniform cross section. Expression (4.2) would have to be modified to include the beams with variable cross sections like the ones show in fig. 4. The modification is in fact quite simple: one would only have to replace the first terms in (4.2) by a quadratic form with z-dependent coefficients. In the case of a helical beam we would take the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \equiv A \equiv A(z) = R^{-1}\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} R,$$

where $R(z)$ is a rotation matrix $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ by z-dependent angle $\alpha = \text{const} \cdot z$.

Remark 4.3. To incorporate torsional deformations of the beam, we could introduce the torsion angle $\alpha(z; t)$, which is the angle formed between the x-axis and the projection onto the xy-plane of the segment rigidly connected to the beam so that in the unperturbed position of the beam it is attached at $(0, 0, z)^{\mathrm{T}}$ and is parallel to the x-axis. Thus the variable $\alpha(z; t)$ describes a normal bundle of the beam. Potential energy of the beam is given by

$$V = \tfrac{1}{2}\int_0^l \left(\langle R^{-1}\mu R v'', v'' \rangle + \mu_3(u_3')^2 + \mu_4(\alpha')^2 \right) \mathrm{d}z,$$

where $\mu = \operatorname{diag}(\mu_1, \mu_2)$, μ_4 is the torsional elasticity coefficient, and $v = (u_1, u_2)$.

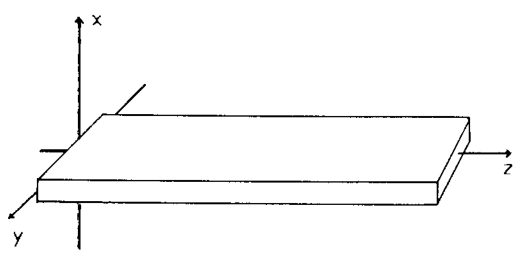

Fig. 3. A beam with elastic coefficients $\mu_1 < \mu_2$.

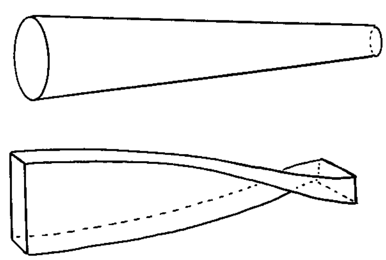

Fig. 4. Beams with variable cross section.

A slightly more subtle remark regarding our model with potential energy defined by (4.2) is that it does not involve potential terms incorporating tensile forces into the model. Our equations of motion (4.4)–(4.6) are somewhat different from those that would be obtained using one of the so-called 'geometrically exact' beam theories, and in particular they do not include terms corresponding to the *dynamical stiffening* noted by Simo and Vu-Quoc ([15]). While such terms may be important in analyzing certain transient regimes, we do not expect that they will significantly change the qualitative picture that emerges from the analysis carried out in the next two sections. Thus to keep our discussion maximally lucid, we eschew the complexity involved in more detailed structural mechanical models, and we refer to the work of Antman and Nachman [2] and Simo and Vu-Quoc [15] for further analysis of the structural mechanics of beams.

The following theorem is a straightforward consequence of the results presented in the previous two sections:

Theorem 4.1. Given the system depicted in fig. 2 and described above with kinetic energy (4.1), potential energy (4.2) and dissipation function (4.3), the equations of motion are given by

$$Da + \left(\mathsf{m}_b\bar{c} + \int_0^l u\right) \times Y^{-1}\ddot{y} = 0, \tag{4.4}$$

$$D^2u + \mu\,\partial u + k\,\partial\dot{u} + Y^{-1}\ddot{y} = 0, \tag{4.5}$$

$$\frac{d^2}{dt^2}(\mathsf{m}_b y + Y\bar{c}) + \int_0^l YD^2u + \ddot{y}\,dz = 0, \tag{4.6}$$

where the quantities in these equations are given as follows. $D(\ ) = d(\ )/dt + \omega \times (\ )$, $a(t) = I\omega(t) + \int_0^l u \times (Du)\,dz$, I is the inertia tensor in the body frame defined above, m_b is the mass of the rigid body component, $\dot{Y} = YS^{-1}(\omega)$ (with $S(\cdot)$ is as defined as in lemma 2.1), and ∂ is the differential operator defined by $\partial = (\partial_z^4, \partial_z^4, -\partial_z^2)$, $\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ and $k = \mathrm{diag}(k_1, k_2, k_3)$.

Eqs. (4.4)–(4.6) are equivalent to

$$I\dot{\omega} + \omega \times I\omega + \int_0^l u \times \big[u_{tt} + \dot{\omega} \times u + 2\omega \times u_t + \omega \times (\omega \times u) + Y^{-1}\ddot{y}\big]\,dz + \mathsf{m}_b\bar{c} \times Y^{-1}\ddot{y} = 0, \tag{4.7}$$

$$u_{tt} + \dot{\omega} \times u + 2\omega \times u_t + \omega \times (\omega \times u) + \mu\,\partial u + k\,\partial u_t + Y^{-1}\ddot{y} = 0, \tag{4.8}$$

$$\mathsf{m}y + Y\left(\int_0^l u\,dz + \mathsf{m}_b\bar{c}\right) = c' + c''t, \tag{4.9}$$

where m_b denotes the mass of the rigid body component, and m denotes the total mass of the body–beam system. Since the mass density of the beam has been normalized to be 1, we find $\mathsf{m} = l + \mathsf{m}_b$. (4.7) can be further rewritten, using (4.8), as

$$I\dot{\omega} + \omega \times I\omega - \int_0^l u \times [\mu\,\partial u + c\,\partial\dot{u}]\,dz + \mathsf{m}_b\bar{c} \times Y^{-1}\ddot{y} = 0. \tag{4.7$'$}$$

One easily checks that the total angular momentum $\int (y + Yu) \times d(y + Yu)\,dt\,dm$ is conserved, either using Noether's theorem or by direct computation.

Remark 4.4. If the center of mass of the system is at rest with respect to the space frame, we may assume that $c' = c'' = 0$.

We indicate the physical meaning of the terms in the equations (4.7)–(4.9). The sum of the first two terms in (4.7) is interpreted as the rate of change of the rigid body's angular momentum. The second term in the brackets gives the inertial force on the beam due to the body's angular acceleration, the third term is the Coriolis force, the fourth is the centrifugal force and the last term is the D'Alembert force. Thus eq. (4.7) expresses the conservation of the total angular momentum in space expressed in the body's coordinate system. Eq. (4.8) is just Newton's law for the beam expressed in the non-inertial body frame (the D'Alembert principle)—it accounts for various inertial forces. Eq. (4.9) expresses the conservation of the linear momentum of the whole system. It is important at this point to make the following remark:

Remark 4.5. There is an apparent paradox associated with eq. (4.7)′: one might expect to be able to express the integral term in terms of u and its derivatives at $z = 0$, since *the body feels the beam only through the attachment point*. As it turns out, this expectation is not met. In formulating our continuum mechanical model of the beam, we have neglected certain effects such as torsional deformations. The implicit rigidity in our model leads to this nonlocal character in the equation.

5. Asymptotic dynamics of a rotating elastic structure

In this section we begin an analysis of the asymptotic behavior of the body-beam structure described in the preceding section. For finite dimensional dissipative mechanical systems, LaSalle's invariance principle [10] can be used to show that states asymptotically approach a minimal invariant subset of the zero set of the dissipation function discussed in section 3. In section 6, it is shown that this type of analysis may be extended to certain infinite dimensional systems with features in common with our body–beam model described by eqs. (4.4)–(4.6). While we do not prove that all solutions to (4.4)–(4.6) tend to the zero set of the dissipation function D as defined by equation (4.3), we do offer a more or less complete characterization of the set of asymptotic equilibrium states (by which we mean the set of solutions to (4.4)–(4.6) for which the equality $D = 0$ also holds). We prove, in particular, that in asymptotic steady state the beam displacement function $u(\cdot, \cdot)$ does not depend on t. Moreover, it is shown that asymptotic equilibrium angular velocities are constant vectors parallel to the principal axes of the steady state inertia tensor. This means that the motion of the system is a pure rotation with no precession. This is the content of the following theorem:

Theorem 5.1. Solutions of (4.7)–(4.9) which are asymptotic equilibria (i.e. solutions which also satisfy $D = 0$ with D as defined in (4.3)) have the following properties:

(i) there is no dependence on the time variable t in the beam function: $u(z, t) = u_{\infty}(z)$;

(ii) the angular velocity ω is a constant ω_{∞};

(iii) the equilibrium angular momentum is a constant, $\mathrm{a}_{\infty} = J_{\infty}\omega_{\infty}$, with the equilibrium inertia tensor of the combined body–beam system given by

$$J_{\infty} = I + \int_0^l u^{\mathrm{T}} u E - u u^{\mathrm{T}} \, \mathrm{d}z - \mathrm{m}\left(C_m^{\mathrm{T}} C_m E - C_m C_m^{\mathrm{T}}\right), \tag{5.1}$$

where E = the identity matrix and $C_m = (1/\mathrm{m})(\mathrm{m_b}\bar{c} + \int_0^l u \, \mathrm{d}z)$ is the center of mass of the body–beam system (expressed in the body frame).

(iv) equilibrium rotations are aligned with a principal axis of the equilibrium inertia tensor, and thus

$$J_{\infty}\omega_{\infty} = \lambda\omega_{\infty}, \tag{5.2}$$

where λ is an eigenvalue of J_{∞}.

Proof. To prove (i), note that for asymptotic equilibria, $\int k_1(\dot{u}_1'')^2 + k_2(\dot{u}_2'')^2 + k_3(\dot{u}_3')^2 \, \mathrm{d}z = 0$. Hence, $\dot{u}_1'' = \dot{u}_2'' = \dot{u}_3' \equiv 0$, and the result follows as a consequence of the boundary conditions.

Using the time independence of u, we show that corresponding values of ω are constant. Since $u_t \equiv 0$, eq. (4.8) may be rewritten

$$\dot{\omega} \times u + \omega \times (\omega \times u) + \mu \, \partial u + Y^{-1}\ddot{y} = 0.$$

Differentiating with respect to z at $z = 0$ and using the boundary condition $\partial u / \partial z = k = (k_1, k_2, k_3)^{\mathrm{T}} = (0, 0, 1)^{\mathrm{T}}$ at $t = 0$, we obtain

$$\dot{\omega} \times k + \omega \times (\omega \times k) + \mu \, \partial u_z(0) = 0.$$

Denoting the last term by $(c_1, c_2, c_3)^T$, rewrite this as

$$\begin{pmatrix} \dot\omega_1 - \omega_2\omega_3 \\ \dot\omega_2 + \omega_1\omega_3 \\ \omega_1^2 + \omega_2^2 \end{pmatrix} = \begin{pmatrix} c_2 \\ -c_1 \\ c_3 \end{pmatrix}. \tag{5.3}$$

Multiplying the first two components by ω_1 and ω_2 respectively and adding we obtain

$$c_2\omega_1 - c_1\omega_2 = \dot\omega_1\omega_1 + \dot\omega_2\omega_2. \tag{5.4}$$

From the last component of (5.3) it follows that the right-hand side is zero, and then (5.4) together with the last component of (5.3) implies ω_1 and ω_2 are constant, if c_1 and c_2 are not both zero.

To establish (ii) in the case that $c_1 = c_2 = 0$, we shall compare the constraint

$$\omega_1^2 + \omega_2^2 = c_3, \tag{5.5}$$

with the constraints provided by the conservation of kinetic energy

$$\omega^T J_\infty \omega = T \tag{5.6}$$

and the conservation of magnitude of angular momentum

$$\|J_\infty \omega\|^2 = M^2. \tag{5.7}$$

If these three polynomial equations have finitely many solutions, then one of them will be the desired time independent ω_∞. It thus remains to investigate the case that the algebraic set specified by these equations has a component of positive dimension. To express (5.5)–(5.7) in terms of a principal axis coordinate system associated with J_∞, we choose an orthogonal matrix U such that $\bar\omega = U\omega$ satisfies

$$\alpha_1\bar\omega_1^2 + \alpha_{12}\bar\omega_1\bar\omega_2 + \alpha_{13}\bar\omega_1\bar\omega_3 + \alpha_2\bar\omega_2^2 + \alpha_{23}\bar\omega_2\bar\omega_3 + \alpha_3\bar\omega_3^2 = c_3, \tag{5.5$'$}$$

$$j_1\bar\omega_1^2 + j_2\bar\omega_2^2 + j_3\bar\omega_3^2 = T, \tag{5.6$'$}$$

$$j_1^2\bar\omega_1^2 + j_2^2\bar\omega_2^2 + j_3^2\bar\omega_3^2 = M^2. \tag{5.7$'$}$$

Any algebraic subset of (5.6)′, (5.7)′ having positive dimension must be symmetric with respect to the origin. Hence we may assume $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$. Moreover, because the diagonal quadratic forms (5.5) and (5.5)′ are related by a similarity transformation, we may assume without loss of generality, that $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 0$. If

$$\det \begin{vmatrix} 1 & 1 & 0 \\ j_1 & j_2 & j_3 \\ j_1^2 & j_2^2 & j_3^2 \end{vmatrix} \neq 0$$

then each triple of values c_3, T, M^2 determines a set of values ω_1^2, ω_2^2 and ω_3^2 uniquely. Thus, as above, ω_∞ must be constant, establishing (ii). If the determinant is zero, either $j_1 = j_2 = j_3$, in which case ω_∞ must be constant, or else there must be constants λ_1, λ_2 such that

$$\lambda_1 \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} + \lambda_2 \begin{pmatrix} j_1^2 \\ j_2^2 \\ j_3^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Neither λ_1 nor λ_2 can be zero. The last component implies that $\lambda_1 + \lambda_2 j_3 = 0$, which allows us to eliminate λ_1 from this system of equations. Both j_1 and j_2 satisfy the polynomial equation $-\lambda_2 j j_3 + \lambda_2 j^2 - 1 = 0$. Either $j_1 = j_2$ or else $j_1 + j_2 = j_3$. The second possibility is ruled out by physical considerations. (It must always be the case that $j_3 < j_1 + j_2$.) Hence, $j_1 = j_2$ and the equations of motion are

$$\dot\omega_1 = \frac{j_1 - j_3}{j_1}\omega_2\omega_3,$$

$$\dot\omega_2 = \frac{j_3 - j_1}{j_1}\omega_1\omega_3,$$

$$\dot\omega_3 = 0.$$

Since $j_3 \neq 0$, these equations are not consistent with (5.3) unless $\omega_\infty = \text{const}$. This concludes the proof of (ii).

(iii) follows from a direct computation using (4.4) and (4.6). It may also be obtained as a direct consequence of the parallel axis theorem.

(iv) follows since ω_∞ must satisfy $\omega_\infty \times J_\infty \omega_\infty = 0$. ∎

Corollary 5.1. The asymptotic equilibrium beam function $u_\infty(\cdot)$ and the asymptotic equilibrium angular velocity ω_∞ are related by eqs. (5.1), (5.2) together with the fourth order system of ordinary differential equations

$$\mu\, \partial u = -\Omega_\infty^2 (u - C_m), \tag{5.8}$$

where c_m is the center of mass of the body–beam system in the body frame, $u = (u_1, u_2, u_3 + z)$, and $S\Omega_\infty = \omega_\infty$, and where the boundary conditions are as prescribed in section 4.

Proof. In light of theorem 5.1, eq. (4.8) may be rewritten as

$$\mu\, \partial u = -\Omega_\infty^2 u - Y^{-1}\ddot{y}.$$

From eq. (4.9) (in which $c' = c'' = 0$) we see that $y(t) = -Y(t)C_m$. Moreover, since Y satisfies $\dot{Y}(t) = Y(t)\Omega_\infty$, we have that $Y(t) = Y_0 e^{\Omega_\infty t}$. Then $\ddot{y} = -Y(t)\Omega_\infty^2 C_m$ and $Y^{-1}\ddot{y} = -\Omega_\infty^2 C_m$, proving the corollary. ∎

Remark 5.1. (5.8) prescribes a nonlinear boundary value problem since Ω_∞ depends on $u_\infty(\cdot)$ through eqs. (5.1) and (5.2). Some idea of the complexity involved in explicitly determining $u_\infty(\cdot)$ may be gleaned from our solution to the model problem described by eqs. (6.1)–(6.3) in the following section.

Remark 5.2. A detailed analysis of the way in which the dissipative dynamical system (4.4)–(4.6) evolves toward steady state is beyond the scope of this paper. Nevertheless, a heuristic description which makes contact with the classical theory of rigid bodies may be given as follows. For a rotating rigid body there are two well known conserved quantities: the kinetic energy E, and the magnitude of angular momentum $|M|$. If M represents the angular momentum vector with respect to the body frame, then the energy $E = \frac{1}{2}M^{\mathrm{T}}I^{-1}M$ is a quadratic form in M with constant coefficients and the conservation of energy confines M to an ellipsoid, resulting in fig. 5a. If a dissipative elastic appendage is present then M, while still confined to a sphere, will move to smaller and smaller energy ellipsoids, as indicated in fig. 5b. For the purpose of this heuristic discussion we ignore the infinite-dimensionality introduced by the beam.

It should be noted that the *basins of the two sinks S_1 and S_2 in* fig. 5b *are interlaced*, and *it is difficult to predict which sink of M will appear if $M(0)$ is near the maximum energy point N*. This somewhat delicate phenomenon is said to have been overlooked in the design of the Explorer II mission. The satellite oriented itself along the proper axis, but in the direction opposite to the desired one.

It must also be noted that the picture in fig. 5 is valid for moderate $|M|$ only, and it undergoes bifurcations as $|M|$ is increased. These bifurcations are due to buckling of the beam at higher angular velocities. Such a phenomenon is analyzed in complete detail in a model problem in the next section.

6. Bifurcations, stability and dissipation in a model problem

In this section we illustrate two interesting phenomena—bifurcations and the trend to the predetermined angular velocities in the simplest possible setting. To bring the underlying phenomena into focus, we look at a simplified model, thereby minimizing technical details while retaining several interesting features. We will see that the example considered below is an infinite-dimensional Morse system. It is similar to the general body-beam system considered in the preceding sections. We also point out an unexpected connection between this problem and that of studying bound states of a nonlinear Schrödinger equation, cf. [14].

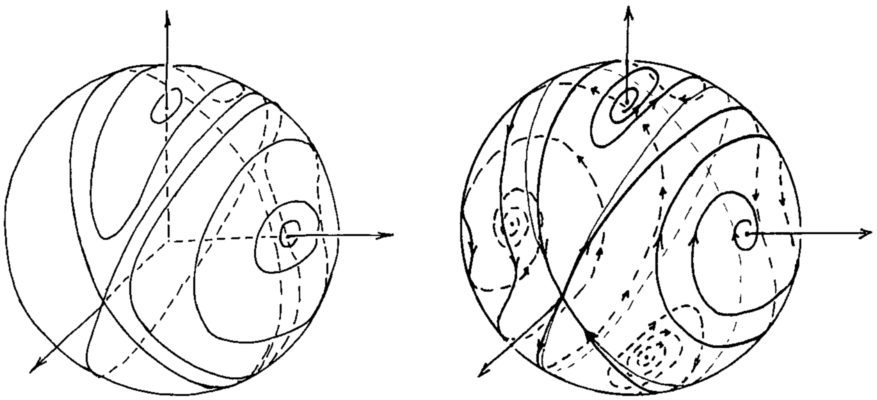

Fig. 5. Momentum spheres in the conservative and dissipative cases.

Fig. 6 shows our model consisting of a disk with a beam attached to its center and perpendicular to the disk's plane.

The disk is constrained to rotate around the z-axis without friction, so that the angular momentum of the system is conserved. The beam is constrained to the z–u-plane, and all the deflections are parallel to the u-axis. Equations of motion of the beam, including internal damping, are

$$\rho u_{tt} + \mu u_{zzzz} + k u_{zzzzt} = \omega^2 u, \tag{6.1}$$

$$\omega\left(\int_0^1 \mu u^2 \, \mathrm{d}z + I\right) = M, \tag{6.2}$$

$$u(0,t) = u_z(0,t) = u_{zz}(1,t) = u_{zzz}(1,t) = 0, \tag{6.3}$$

where μ, k characterize elasticity and damping respectively, and eq. (6.2) expresses conservation of angular momentum, I being the disk's moment

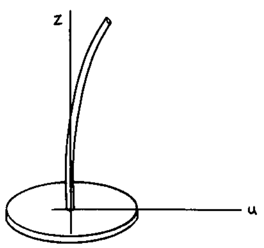

Fig. 6. A model problem with 1 rotational d.o.f.

of inertia. From now on we set $\rho = \mu = 1$ to simplify notation. *Steady-state solutions* will satisfy the ordinary differential equation

$$-u_{zzzz} + \omega^2 u = 0. \tag{6.4}$$

The steady-state equations define a nonlinear eigenvalue problem with M as a parameter, the nonlinearity lying in the u-dependence of ω in (6.2).

Remark 6.1. The stationary problem (6.4)–(6.3) admits a revealing variational formulation: its solutions are the critical points of total energy

$$E = \tfrac{1}{2} I \omega^2 + \int_0^1 \left(u_{zz}^2 + \omega^2 u + u_t^2\right) \mathrm{d}z \tag{6.5}$$

when the angular momentum M is fixed, as one easily checks. This has a simple physical explanation: As the beam vibrates energy is dissipated. Ultimately the vibrations are damped out, and the total energy of the system is extremized. Actually, for some exceptional (in a sense which we make precise in theorem 6.1) initial conditions the limiting energy value will be not minimal, but rather critical. (See theorem 6.1 and fig. 5.) We note also that

$$\dot{E} = -\int_0^1 (u_{zzt})^2 \, \mathrm{d}z. \tag{6.6}$$

Remark 6.2 *on stability.* The same variational formulation suggests a stability criterion for the dynamical system (6.1)–(6.3). Namely, consider the space S of all triples $(u, \dot{u}, \omega)$ with angular momentum M, i.e. satisfying (6.2). If the stationary solution $x_0 = (u, 0, \omega)$ minimizes total energy (6.5) on S, then this stationary solution is stable, even in the undamped case. This means that for all initial data $(u, u_t, \omega)_{t=0}$ in S sufficiently close to the minimal $(u, 0, \omega)$ (in the energy metric) the solution will stay close to the minimizing solution x_0 for all time. This is quite similar to the standard minimal energy criterion of Lagrange; here, however, the energy is minimal only subject to the angular momentum being constrained. This minimality is guaranteed if the following condition holds: $E_{xx} + \lambda M_{xx} > 0$ on the tangent space to $M = \text{const.}$ at the point x_0, where λ is the Lagrange multiplier: $M_x = \lambda E_x$.

This is the idea behind the energy–Casimir method, which is the second derivative test in the presence of the constraints and which is applicable in the infinite-dimensional settings. This method has been used to establish stability in a variety of problems (see, e.g., the work and references in Krishnaprasad and Marsden [8]).

A complete picture of the behavior of our model system, in particular of the global phase portrait and its bifurcations, is given by theorems 6.1 and 6.2 below. A perhaps surprising consequence of this result is that *the system selects a discrete set of angular velocities for its stationary rotations independent of* the angular momentum M. Another aspect of this result is that our model is an *infinite-dimensional Morse system.*

To state our first result we combine (6.1) and (6.2) into a single nonlinear equation

$$u_{tt} + u_{zzzz} + k u_{zzzzt} = \omega^2(u) u,$$

where $\omega(u) = M/(\|u\|^2 + I)$, $\|u\|^2 = \int_0^1 u^2 \, \mathrm{d}z$. This can be rewritten as a system

$$\dot{\omega} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\partial^4 u - \partial^4 v + \omega^2(u) u \end{pmatrix} \equiv F(\omega). \tag{6.7}$$

Theorem 6.1. *Qualitative behavior of the system.* Assume that $k > 0$, i.e. the damping is present.

There exists an infinite sequence $0 < \omega_1 < \omega_2 < \cdots \to \infty$ of preferred angular velocities associated with the problem (6.7) with boundary conditions (6.3) such that if the angular momentum M lies in the interval $(M_k, M_{k+1}) = (I\omega_k, I\omega_{k+1})$, the problem (6.7), (6.3) has k distinct nontrivial stationary solutions $(u, v) = (u_i(z), 0)$, $1 \le i \le k$, and a trivial solution $(u, v) = (0, 0)$, with angular velocities $\omega(u_i) \equiv M/(I + \int_0^1 u_i^2 \, \mathrm{d}z) = \omega_i$, $1 \le i \le k$ and $\omega(0) = M/I$. These solutions are depicted in fig. 7.

The solution with the smallest angular velocity is dynamically stable, i.e. any solution of (6.7), (6.3) with nearby initial conditions remains close to it for all time, in the L^2-norm given by

$$\|(u, \dot{u})\| = \|(u, v)\| = \int_0^1 (u^2 + v^2) \, \mathrm{d}z.$$

Stationary solutions with higher angular velocities are unstable, and furthermore, ith solution (in the order of increasing angular velocity) $(u, \dot{v}) \equiv (u_i, 0)$ has $(i-1)$-dimensional unstable manifold; this dimension coincides with the number of zeroes of u_i. In particular, if $M < M_1 = I\omega_1$, only one mode $u \equiv 0$ is present and is stable. As M crosses the bifurcation values M_i, the system undergoes a series of pitchfork bifurcations shown in fig. 8.

There have been several recent studies of the evolution of solutions to infinite dimensional systems toward equilibria (see Dafermos [4] and references therein). Theorem 6.2 below shows that under mild (physically reasonable) assumptions solutions of our systems approach the equilibrium states described above.

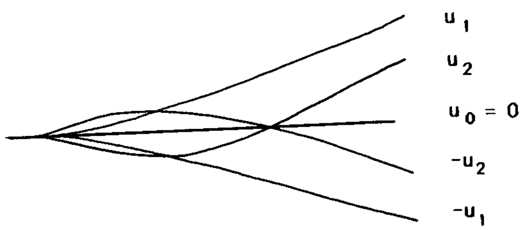

Fig. 7. Solutions to (6.1)–(6.3).

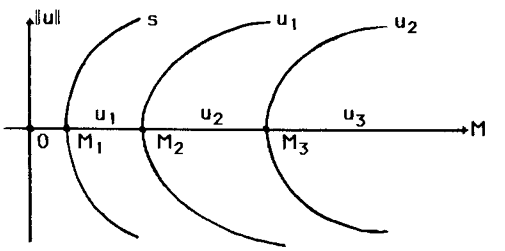

Fig. 8. S denotes stable equilibria, and $u_1, u_2, \ldots$ denote unstable equilibria with $1-, 2-, \ldots$ dimensional unstable manifolds. The numbers and L^2-norm of equilibria depend on the parameter M.

Theorem 6.2. Fix $M>0$ and let $k>0$, say $k=1$. Let $H^{(n)}(0,1)$ denote the Sobolev space of real valued functions defined on $[0,1]$ whose derivatives up to order n are square integrable. Any solution $\omega(t)$ of (6.7) with $u_0(z), v_0(z) \in H^{(12)}(0,1)$ tends in the L^2-norm as $t \to \infty$ to one of the stationary solutions from theorem 6.1 (see fig. 9).

Remark 6.3. The rather detailed results of this section are derived under assumptions of smoothness in the spatial variable on the functions $u(t,x)$ defining beam configurations. No claim is made that these are the weakest possible assumptions under which our analysis could be carried out. On the other hand, it is clear that strong smoothness conditions such as these are physically reasonable.

Proof of theorem 6.1. 1) *Existence of* $k+1$ *solutions for* $M_k < M < M_{k+1}$. Stationary solutions satisfy the nonlinear boundary value problem (6.3)–(6.4). There exists a sequence $0<\omega_1<\omega_2<$

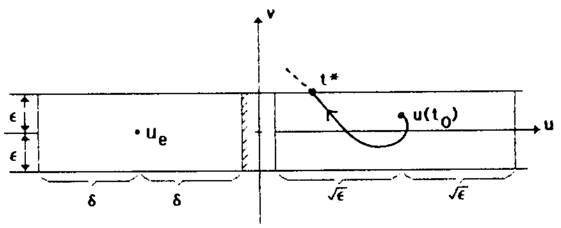

Fig. 10. Solutions which remain ϵ-close to $\{v=0\}$ must be ϵ-close to a stationary point.

$\ldots$ of angular velocities for which this problem has nontrivial solutions (the eigenfunctions of $\partial^4/\partial z^4$ with boundary conditions (6.3)) which we denote by $c_i e_i$, c_i arbitrary and $\|e_i(z)\|_{L^2}=1$. Define $M_j = I\omega_j$*. Freedom of choice of c_i is used to satisfy the angular momentum constraint (6.2), which becomes

$$\omega_i\left(c_i^2+I\right)=M.$$

This can be solved for c_i if and only if $M > I\omega_i \equiv M_i$; setting $u_i = c_i e_i$ proves the existence of k solutions if $M > M_k$. One additional solution is obtained by setting $u_{k+1}\equiv 0$, $\omega_{k+1}=M/I$.

2) *Stability of stationary modes.* We restrict ourselves to the undamped ($k=0$) case, as the dissipation does not affect stability of *stationary* modes. The governing equations can be written as

$$u_{tt} = -u_{zzzz} + \left(\frac{M}{\|u\|^2+I}\right)^2 u \overset{\text{def}}{=} A(u). \tag{6.8}$$

*M_j is the total angular momentum when the beam is undeflected, since the straight beam's ($u\equiv 0$) contribution to the angular momentum is zero.

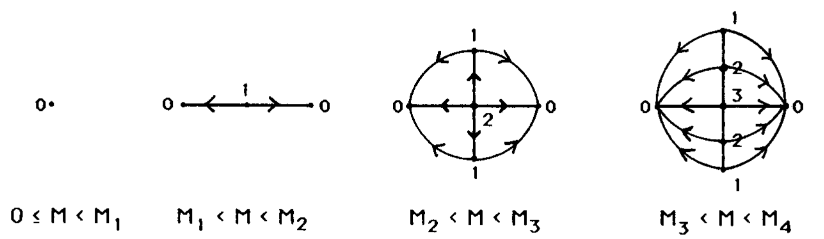

Fig. 9. Global phase portrait of the system (6.7), (6.3). Indices show the dimension of the unstable manifolds, which are depicted by arrows.

The dimension of the unstable manifold of the mode $u_i(z)$ is given by the number of positive eigenvalues of the linearization of the operator $A(u)$ at $u = u_i$. This linearization is given by the Gateaux derivative $A'(u)$:

$$A'(u)v = -\partial^4 v + \left(\frac{M}{\|u\|^2 + I}\right)^2 v - \frac{4M^2}{(\|u\|^2 + I)^3}(u, v)u,$$

where $\partial = \partial/\partial z$ and $(u, v) = \int_0^1 u(z)v(z)\,\mathrm{d}z$.

At $u = u_i$ we have

$$B_v \equiv A'(u_i)v = -\partial^4 v + \omega_i^2 v - 4\omega_i^3 M^{-1}(u_i, v)u_i.$$

The last term in this expression is a scalar multiple of the orthogonal projection of v onto u_i. Comparing B with the operator C given by

$$Cv = -\partial^4 v + \omega_i^2 v,$$

we see that the spectrum of B can be obtained from that of C by replacing the eigenvalue 0 of C corresponding to the eigenfunction u_i by $-4\omega_i^3 M^{-1}\|u_i\|^2$. Thus the eigenvalues of B are

$$\omega_i^2 - \omega_1^2, \omega_i^2 - \omega_2^2, \dots, \omega_i^2 - \omega_{i-1}^2, -4\omega_i^3 M^{-1}\|u_i\|^2, \omega_i^2 - \omega_{i+1}^2, \dots .$$

of which precisely $i-1$ are positive. This shows that the dimension of the unstable manifold at the stationary solution u_i is $(i-1)$. ■

Proof of theorem 6.2 consists of two steps:

(1) Showing that for any $\epsilon > 0$ the solution $w(t)$ enters and stays in an ϵ-neighborhood (in L^2 norm) of the zero set of the dissipation function. This set is $\{(u, v)\colon \int_0^1 v_{xx}^2\,\mathrm{d}x = 0\} \equiv \{(u, v) : (u(x), 0), u \in H^{(12)}\}$; see (6.6).

(2) Showing that if $w(t)$ stays in an ϵ-neighborhood of the zero set of $\{v \equiv 0\}$, then w is ϵ-close to one of the stationary points of the flow, fig. 10. ■

Proof of step 1. Pick $\epsilon > 0$, and consider two neighborhoods of $\{v \equiv 0\}$: $\mathcal{N}_\epsilon = \{(u, v) : \|v\|_{L^2} < \epsilon\}$ and $\mathcal{N}_{\epsilon/2}$. Our solution $w(t)$ enters $\mathcal{N}_{\epsilon/2}$ for some $t > 0$ with necessity: otherwise for all t large enough we would have

$$\dot{E} = -\int_0^1 u_{zzt}^2\,\mathrm{d}z = -\int_0^1 v_{zz}^2\,\mathrm{d}z \le -\lambda_0 \int_0^1 v^2\,\mathrm{d}z \le -\lambda_0\left(\frac{\epsilon}{2}\right)^2, \tag{6.9}$$

resulting in $E(t) \to -\infty$, which is a contradiction. Here λ_0 is the smallest eigenvalue of $\partial^4 = (\partial^4/\partial z^4)$ with boundary conditions (6.3).

Thus to prove that $w(t)$ stays in $\mathcal{N}_\epsilon$ forever after some $T > 0$ it remains only to exclude the possibility of infinitely many trips between $\mathcal{N}_{\epsilon/2}$ and the exterior of $\mathcal{N}_\epsilon$, see fig. 10. We will do so by showing that each trip results in a loss of energy at least $\Delta E > 0$ depending only on ϵ and $w(0)$ at each crossing, so that $w(t)$ can afford only finite number of trips.

Let t_1, t_2 be two consecutive crossing times of the boundaries of $\mathcal{N}_\epsilon$ and $\mathcal{N}_{\epsilon/2}$. We have

$$|E(t_2) - E(t_1)| = \left|\int_{t_1}^{t_2}\int_0^1 v_{zz}^2\,\mathrm{d}z\right| \ge |t_2 - t_1|\lambda_0\left(\frac{\epsilon}{2}\right)^2, \tag{6.10}$$

where we have used the fact that $\int_0^1 v^2\,\mathrm{d}z \ge (\epsilon/2)^2$ and the Poincaré type estimate as in (6.9). It remains only to provide a lower bound on the trip time $|t_2 - t_1|$; it is provided by the upper bound on the velocity $\dot{w}(t)$ which is the result of the smoothness assumption. ■

Lemma 6.1. Any solution $w(t) = (u(z, t), v(z, t))$ of (6.7) with $u_0(z), v_0(z) \in H^{(12)}(0,1)$ has velocity bounded in the L^2 norm: i.e. there exists $C = C(u_0, v_0) > 0$ such that for all $t \ge 0$

$$\|F(w(t))\|^2 = \|v\|_{L^2}^2 + \|-\partial^4 u - \partial^4 v + \omega^2(u)u\|_{L^2}^2 < C^2.$$

Proof of this lemma is given in the appendix.

By the choice of t_1, t_2 and using lemma 6.1, we obtain

$$\frac{\epsilon}{2} \leq \| w(t_2) - w(t_1) \| = \left\| \int_{t_1}^{t_2} F(w(s))\,\mathrm{d}s \right\|$$

$$\leq |t_2 - t_1| \sup_{t \geq 0} \|F\| \leq C|t_2 - t_1|,$$

implying

$$|t_2 - t_1| \geq \frac{\epsilon}{2C},$$

which together with (6.10) proves that $w(t)$ stays in $\mathscr{N}_\epsilon$ for all $t \geq T(\epsilon)$.

Proof of step 2. We show now that if $w(t)$ stays in $\mathscr{N}_\epsilon$ for all $t \geq T(\epsilon)$ then it must tend to an equilibrium point of the flow (6.7). More precisely, we will show that for every $\delta > 0$ there exists an $\epsilon > 0$ such that if $\|v\| \leq \epsilon$ for all $t \geq T(\epsilon)$ then for some equilibrium solution $w_\mathrm{e} = (u_\mathrm{e}, 0)$ we have $\|u_\mathrm{e} - u\|_{\mathrm{L}^2} \leq \delta$ for all $t \geq T(\epsilon)$. Our strategy is to show that if a solution is $v(t)$ *not* close to an equilibrium, then its velocity $v(t)$ must grow, thus taking it outside the neighborhood $\mathscr{N}_\epsilon$, leading to a contradiction.

Assume the contrary: There is some $\delta > 0$ such that for every $\epsilon > 0$ there is a $t_0 = t_0(\epsilon) \geq T(\epsilon)$ such that although

$$\| v(t) \| \leq \epsilon \quad \text{for all } t \geq T(\epsilon), \tag{6.11}$$

we have

$$\| u(t_0) - u_\mathrm{e} \| \geq \delta, \tag{6.12}$$

for every equilibrium u_e. The idea is to show that being far from any equilibrium causes an increase in velocity thus causing $\|v\| > \epsilon$ (a contradiction). The details are as follows.

(6.12) implies for $f(u) \equiv -\partial^4 u + \omega^2(u) u$:

$$\| f(u(t_0)) \| \geq \lambda \| u(t_0) - u_\mathrm{e} \| \geq \lambda\delta, \tag{6.13}$$

for some $\lambda > 0$ independent of δ, if we choose δ small enough (which we do).

We have (cf (6.7))

$$\| v(t) - v(t_0) \|_{\mathrm{L}^2} \geq \left\| \int_{t_0}^{t} f(u(s))\,\mathrm{d}s \right\| - \left\| \int_{t_0}^{t} \partial^4 v(s)\,\mathrm{d}s \right\|. \tag{6.14}$$

From the proof of lemma 6.1 it is clear that there exists $C > 0$ such that $\|\partial^4 v\|_{\mathrm{L}^2} \leq C_\epsilon$ for t large enough; without loss of generality we assume that this already holds for $t \geq t_0$.

We have from (6.14)

$$\| v(t) \| \geq \left\| \int_{t_0}^{t} f(u(s))\,\mathrm{d}s \right\| - \epsilon(1 + C)(t - t_0) \quad \text{for all } t \geq t_0. \tag{6.15}$$

Furthermore, if t', t'' are sufficiently large, and if

$$\| u(t'') - u(t') \| \leq \sqrt{\epsilon}$$

holds, then

$$\| f(u(t'')) - f(u(t')) \| \leq \tfrac{1}{2} \| f(u(t')) \|.$$

This can be seen from the proof of lemma 6.1 which shows that all the harmonics of sufficiently high order decay exponentially, with $w(t)$ tending to a finite-dimensional subspace of $\mathrm{L}^2 \times \mathrm{L}^2$. Applying this remark to (6.15) with $t' = t_0$, $t'' = t$, we obtain the final lower bound on v, using (6.13),

$$\begin{aligned} \| v(t) \| &\geq \tfrac{1}{2}(t - t_0) \| f(u(t_0)) \| - \epsilon(1 + C)(t - t_0) \\ &\geq (t - t_0)\left[\tfrac{1}{2}\lambda\delta - \epsilon(1 + C)\right], \end{aligned}$$

as long as

$$\| u(t) - u(t_0) \| \leq \sqrt{\epsilon}.$$

Using this estimate on the velocity $v(t)$, we will show that $w(t)$ leaves the box $\|v\| \leq \epsilon$, $\|u - u(t_0)\| \leq \sqrt{\epsilon}$ through the "horizontal" boundary, i.e.

$\|v(t^*)\| = \epsilon$ for some first exit time $t^* > t_0$, see fig. 10. This will result in the desired contradiction.

To see that $\|v\| = \epsilon$ is reached first, we note that otherwise

$$\|u(t^*) - u(t_0)\| = \sqrt{\epsilon},$$

and

$$\epsilon(t^* - t_0) \geq \left\| \int_{t_0}^{t^*} v(t)\,\mathrm{d}t \right\|$$
$$= \|u(t^*) - u(t_0)\| = \sqrt{\epsilon},$$

i.e. the time required to reach the *vertical* boundary is large

$$t^* - t_0 \geq \frac{1}{\sqrt{\epsilon}}.$$

Using this in (6.16) – which is valid for $t_0 \leq t \leq t^*$ – we obtain

$$\left\| v\left(t_0 + \frac{1}{\sqrt{\epsilon}}\right) \right\| \geq \frac{1}{\sqrt{\epsilon}} \left[\frac{1}{2}\lambda\delta - \epsilon(1 + C)\right] > \epsilon,$$

which is a desired contradiction, if we pick ϵ sufficiently small. ■

This shows that given any $\delta > 0$ there exists an $\epsilon > 0$ such that if w stays in $\mathcal{N}_\epsilon$, then $u(z, t)$ is less than δ (in $L^2[0,1]$) from an equilibrium solution u_e of (6.8). Since the first step in this proof showed that for ϵ-neighborhood of the zero set $\{v \equiv 0\}$ of the dissipation function, this proves theorem 6.2.

Appendix

Proof of lemma 6.1

The nonlinearity in (6.7) is Lipschitz continuous in $L^2(0,1)$, and thus standard existence results imply that $u(x, t)$ and $v(x, t) \in H^{(12)}(0,1)$.

We may thus expand $u(x, t)$ in the orthonormal basis $\{e_j(x)\}$ of the eigenfunctions of ∂^4 (which is a self-adjoint operator on $L^2(0,1)$) with the boundary conditions (6.3):

$$u(x, t) = \sum_{k=0}^{\infty} a_k(t) e_k(x),$$

which converges in $L^2(0,1)$. System (6.1) is equivalent to a series of ODE's for the amplitudes:

$$\ddot{a}_k + \lambda_k \dot{a}_k + \lambda_k a_k = \omega^2 a_k,$$

$\{\lambda_k\}$ are the eigenvalues of ∂^4, or

$$\begin{cases} \dot{a} = b, \\ \dot{b} = -\lambda b - \lambda a + \omega^2 a, \end{cases} \tag{A.1}$$

where we have dropped the subscript k for the sake of brevity.

It suffices to prove that there exists $C > 0$ such that

$$\|\partial^4 u\|_{L^2}^2 + \|\partial^4 v\|_{L^2}^2 = \sum_{k=0}^{\infty} \lambda_k^2 \big(a_k^2(t) + b_k^2(t)\big) < C \quad \text{for all } t \geq 0. \tag{A.2}$$

We note that the expansions of $\partial^4 u$ and $\partial^4 v$ in the basis $\{e_k(x)\}$ converge in L^2. This shows that $\|F(\omega)\|_{L^2}$ is bounded for all t, since the boundedness of the last term $\omega^2(u)u$ in L^2 is obtained as follows:

$$\|\omega^2(u)u\|_{L^2}^2 = \omega^4(u)\|u\|^2 \leq \left(\frac{M}{I}\right)^4 \|u\|^2$$
$$\leq \left(\frac{M}{I}\right)^4 \frac{1}{\lambda_0^2} \|\partial^4 u\|_{L^2}^2 \leq \left(\frac{M}{I}\right)^4 \lambda_0^{-2} C.$$

To estimate $a_k(t)$, $b_k(t)$ we use the smoothness of the initial conditions to get an upper bound on $a_k(0)$, $b_k(0)$: estimates

$$\|\partial^{12} u_0\|_{L^2}^2 = \sum a_k^2(0)\lambda_k^6 < c,$$
$$\|\partial^{12} v_0\|_{L^2}^2 = \sum b_k^2(0)\lambda_k^6 < c,$$

imply

$$|a_k(0)|, |b_k(0)| < c\lambda_k^{-3}.$$

We will show that for all $t \geq 0$ the estimates

$$|a_k(t)|, |b_k(t)| < c\lambda_k^{-2} \tag{A.3}$$

hold, for k large enough, thus implying (A.2)

$$\sum \lambda_k^2 a_k^2 \leq \sum \lambda_k^2 c^2 \lambda_k^{-4} = c^2 \sum \lambda_k^{-2} < \infty$$

since $\lambda_k \sim k^4$.

To show (A.3) we rewrite (A.1) as

$$\dot{z} = Az + Rz,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\lambda & -\lambda \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & 0 \\ \omega^2 & 0 \end{pmatrix}.$$

Introducing the Lyapunov matrix

$$B = \int_0^\infty e^{A^T s} e^{As} ds, \quad \text{satisfying } A^T B + BA = -I,$$

we obtain

$$\frac{d}{dt}(Bz, z) = -(z, z) + ((R^T B + BR)z, z)$$
$$\leq -(1 - \|R^T B + BR\|)(z, z).$$

An explicit computation gives

$$B = \begin{pmatrix} 1 + \frac{1}{2\lambda} & \frac{1}{2\lambda} \\ \frac{1}{2\lambda} & \frac{1}{2\lambda} + \frac{1}{2\lambda^2} \end{pmatrix}$$

and thus $\|R^T B + BR\| \leq$ "const."$/\lambda$, since $\omega < M/T$.

We have

$$\frac{d}{dt}(Bz, z) \leq -\left(1 - \frac{\text{"const."}}{\lambda}\right)(z, z)$$
$$\leq -\tfrac{1}{2}(z, z),$$

the latter inequality holding for all $\lambda = \lambda_k$ large enough.

Now, the minimal eigenvalue ρ of the positive definite matrix B is estimated from below as

$$\rho > \frac{1}{2\lambda} \quad (\text{for } \lambda \text{ large enough}).$$

We have for all $t > 0$ and for k large enough

$$\frac{1}{2\lambda}(z(t), z(t)) < (Bz(t), z(t))$$
$$\leq (Bz_0, z_0) \leq (z_0, z_0),$$

or

$$a_k^2(t) + b_k^2(t) \leq 2\lambda_k(a_k^2(0) + b_k^2(0))$$
$$\leq 2\lambda_k c^2 \lambda_k^{-6} = 2c^2 \lambda_k^{-5}.$$

This implies (A.3). ∎

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