Telescopes and symplectic mappings

MATHEMATICAL CURIOSITIES
By Mark Levi

Symplectic geometry, tracing its origins to the work of Poincaré on Hamiltonian systems, and currently a very active field, reached a high level of abstraction. Here is a simple concrete example where "symplectic" approach predicts and explains the following physical fact:

Any optical device (e.g. a telescope) which converts a parallel beam to a narrower parallel beam must necessarily magnify objects.[1]

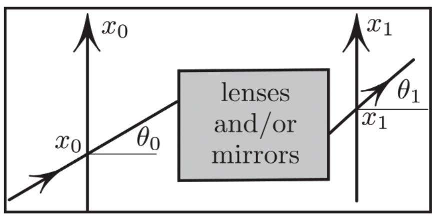

Figure 1: *We place an axis x_0 before the device and another axis x_1 after. The entry data (x_0, α_0) of a ray determine its exit data (x_0, α_1).*

This, as I will show next, is a manifestation of the obvious fact that *if an area preserving a map squeezes in one direction, it must expand in another.*

[1] To maximize simplicity, I minimize the dimension to two.

Any optical device (in two dimensions) – schematically, the black box in Figure 1 – gives rise to a map which assigns to each ray's entry data (x_0, y_0) where $y_0 = sin\,\theta_0$, the corresponding exit data (x_1, y_1) Parallel beams, e.g. CD and C′D′ in Figure 2, correspond to horizontal segments in the xy-plane. Horizontal segments map under φ to horizontal segments; moreover, φ shortens these segments since the device narrows parallel beams, Figure 2.

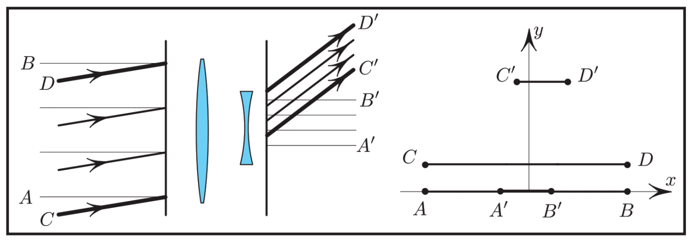

Figure 2: *Left: Beam CD exits as beam C′D′. Right: φ (CD) = C′D′.*

Now φ is area–preserving.[2] And since φ squeezes the rectangle *ABDC* (Figure 3) in the *x*–direction, it must stretch in the *y*–direction. This *y*–stretching means that *the angles between parallel beams are magnified.* But this is precisely what the optical magnification of objects amounts to. For example, the reason a telescope allows us to tell that a distant speck is actually a ship and not a dot is that it increases the angle between two beams, one from the stern and the other from the bow, thus making these beams fall onto *different* "pixels" on our retina.

The proof of area–preservation in the footnote, due to Poincaré, admits a "hands–on" palpable mechanical interpretation, as described in [1]. More on symplectic maps and lenses can be found in the remarkable book [2]. And there are interesting open questions that we do not address here on the relationship between recent results of symplectic geometry to optics.

[2] To see why, consider the travel time T (x_0, x_1) (called the optical distance), and note that $y_0 = -T_{x_0}(x_0, x_1)$, $y_1 = T_{x_1}(x_0, x_1)$, subscripts denoting partial differentiation. Then for a closed curve γ_0 in the (x_0, y_0) plane, parametrized $s \in [0,1]$ we get

$$0 = \int_0^1 \frac{d}{ds} T(x_0(s), x_1(s))ds \; = \int_0^1 (-y_0 x_0' + y_1 x_1')ds$$
$$= -\int_{\gamma_0} y\,dx + \int_{\varphi(\gamma_0)} y\,dx.$$

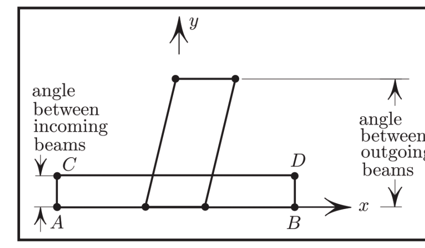

Figure 3: *Narrowing of the beams causes widening of the angles between beams, i.e. the optical magnification.*

References

[1] M. Levi, Classical Mechanics with Calculus of Variations and Optimal Control: an Intuitive Introduction. AMS, March 2014, pp. 46 and 275.

[2] V. Guillemin and S. Sternberg, Symplectic Techniques in Physics. Cambridge University Press, 1984 or later editions.

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