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Constrained Relative Motions in Rotational Mechanics

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Abstract

The dynamical effects of imposing constraints on the relative motions of component parts in a rotating mechanical system or structure are explored. It is noted that various simplifying assumptions in modeling the dynamics of elastic beams imply strain constraints, i.e., that the structure being modeled is rigid in certain directions. In a number of cases, such assumptions predict features in both the equilibrium and dynamic behavior which are qualitatively different from what is seen if the assumptions are relaxed. It is argued that many pitfalls may be avoided by adopting so-called geometrically exact models, and examples from the recent literature are cited to demonstrate the consequences of not doing this. These remarks are brought into focus by a detailed discussion of the nonlinear, nonlocal model of a shear-free, inextensible beam attached to a rotating rigid body. Here it is shown that linearization of the equations of motion about certain relative equilibrium configurations leads to a partial differential equation. Such spatially localized models are not obtained in general, however, and this leaves open questions regarding the way in which the geometry of a complex structure influences computational requirements and the possibility of exploiting parallelism in performing simulations. A general treatment of linearization about implicit solutions to equilibrium equations is presented and it is shown that this approach avoids unintended imposition of constraints on relative motions in the models. Finally, the example of a rotating kinematic chain shows how constraining the relative motions in a rotating mechanical system may destabilize uniformly rotating states.

1. Introduction

There is at present a rapidly growing literature on the mechanics and control of rotating structures and multibody systems. In the present paper, it is shown that many assumptions which are made either implicitly or explicitly in deriving dynamical models of elastic solids (specifically beams) are equivalent to assuming there are kinematic constraints on relative motions of material elements within the solid. It will be shown how such constraints can dramatically affect the qualitative dynamics predicted by the model in the case of a rotating system. In discussing a number of different models of rotating planar thin beams, it is noted that implicit constraints in some cases, such as the Euler-Bernoulli beam and the so-called "shear beam," result in spatially localized descriptions in which the (partial differential) equations of motion do not reflect any direct dynamic coupling between beam elements which are any positive distance from one another. We would not expect the rotational dynamics of beams made of many materials (such as steel or other common metals) to be accurately described by such localized models because centrifugal and other forces acting essentially in the longitudinal direction of the beam will be transmitted instantaneously along the beam due to its relative rigidity in this direction. To examine such effects more closely, we propose and study a model in which it is explicitly assumed that the beam is inextensible and moreover does not undergo shear deformation. The resulting model is a partial integro-differential equation which explicitly captures the nonlinear dynamic coupling between beam elements. To study the linearization of such a model for a rotating beam, we find that it is useful to incorporate a high-order approximation to the equilibrium solutions to the equations of motion. It is shown that this approach produces linearized models which, although they are an approximation, retain essential qualitative features of the geometrically exact nonlinear models. The example of a rotating string will serve to illustrate these ideas.

The point of departure for our discussion is a previous paper [1] which was, in part, devoted to characterizing the steady-state rotations of a rigid body with a flexible beam attachment. We show, in Section 2, that Theorem 5.1 of [1] may be rephrased to give a general characterization of steady-state rotations in a wide class of rotating elastic solids. In particular, under a very general set of assumptions, it is shown that steady state rotations always involve constant angular velocities about a principal axis of the steady-state inertia tensor. If the model of the solid in question exhibits partial rigidity, however, and the deformation is constrained in certain spatial dimensions, we show by example that the relative equilibria have a rather different qualitative character with nonconstant angular velocities (precession) being possible.

Motivated by the large recent literature dealing with the dynamics of rotating beams, we shall devote Section 3 to a discussion of the qualitative features of several (planar) beam theories. Subject to the understanding that one-dimensional continuum theories of thin beams involve implicit modeling assumptions (see [2]), we begin by deriving a "general" model of a planar beam. We then simplify this model in several ways, and discuss how each simplification alters the predicted qualitative dynamics. For instance, the classical Euler-Bernoulli beam model, derived by ignoring terms of order higher than one in the strains, results in a model in which infinitesimal beam segments move vertically in the model coordinate frame. It is noted that this is not unlike the behavior of the "shear beam" recently studied by KRISHNAPRASAD & MARSDEN [3]. There is a resulting implicit dimension of rigidity in the model which we show implies that a rotating rigid body with shear-beam attachment has a discrete set of possible angular velocities in steady-state. (See Theorem 3.3.) An interesting contrast is seen in the model proposed for an inextensible, unshearable beam; here the general model has been simplified by imposing rigidity assumptions explicitly — unlike the Euler-Bernoulli model where rigidity appears to enter as an artifact of the model rather than a planned feature. By making these assumptions conform to the geometry of the problem, we argue that the resulting model accurately portrays the dynamics of thin beams. Motions of individual material elements in the beam are strongly coupled in this case, and this results in the dynamics being given in terms of nonlinear partial differential-integral equations. It is not surprising to find a high degree of nonlinear coupling among system states, but it is of passing interest to note that the other models which are considered (including the so-called shear-beam and the Euler-Bernoulli beam) are all spatially localized in the sense that they are described by partial differential equations. Of the simplified models considered in Section 3, only the inextensible beam model given by the differential-integral equations (18)–(20) appears to be free of spurious qualitative features.

In Section 4, we examine linearized models derived from the nonlinear partial differential-integral equations describing an inextensible, shear-free beam attached to a rotating rigid body. It is shown that linearized models of such a system *are* spatially localized and give rise to partial differential equations precisely when the linearization is about relative equilibrium rotations in which the equilibrium configuration of the beam lies along a ray passing through the center of mass of the rigid body. These localized models correspond to the classical Euler-Bernoulli (or Rayleigh) beam model when there is no underlying rotation of the system defining the relative equilibrium, and they include terms reflecting centrifugal stiffening when steady-state rotation is present. While the recent work of SIMO & VU-QUOC [4] points to the need for incorporating nonlinear strain measures in order to capture the effect of centrifugal stiffening, we shall argue that the mechanism producing this and other important nonlinear effects is of a fundamentally geometric nature. In Section 5 we point out that linearized models will in general capture nonlinear effects such as centrifugal stiffening *provided* the nonlinear models we are linearizing accurately reflect the geometry of relative motions in the system and provided that linearization is carried out in a neighborhood of a true equilibrium solution. In Section 6, we look at the dynamics of a pendulum which is suspended by a universal joint and rotated about an axis aligned with gravity. We compare the dynamics with a rotating simple pendulum (suspended by a one degree of freedom joint) and once again find striking qualitative differences caused by constraints on the relative motions in the latter case.

2. Partially Rigid Bodies: An Example of a Nonholonomically Constrained Rotating System

In BAILLIEUL & LEVI [1], it was shown how the equations of motion for a rotating elastic body could be formulated in terms of a configuration space $\{(Y, y, u)\} = SO(3) \times \mathbb{R}^3 \times C$ where $SO(3)$ denotes the special orthogonal group of 3×3 orthogonal matrices with determinant one and $C = \{u(\cdot, t)\}$ is a suitably defined function space. A typical configuration (Y, y, u) assigns to the body a "body frame" whose orientation and origin are given with respect to an inertially fixed frame of reference by the pair (Y, y). Each member $u(\cdot) \in C$ describes a configuration of the structure with respect to this body frame. More specifically, there is an underlying body $\mathscr{B}$ which, following TRUESDELL & NOLL [5], is a differentiable manifold. Each member $u(\cdot) \in C$ is a homeomorphism mapping $\mathscr{B}$ onto a subset of $\mathbb{R}^3$, thereby defining a configuration of the body, which we choose to be *with respect to the body frame*. The corresponding configuration with respect to inertial space may be expressed in terms of a function $u_I(\cdot)$ defined by the formula

$$u_I(z) = Y \cdot u(z) + y.$$

The configuration changes with time, and a point (or particle) $z \in \mathscr{B}$ occupies a location

$$u_I(z, t) = Y(t) \cdot u(z, t) + y(t)$$

in $\mathbb{R}^3$ at time t. The velocity of this point is given by

$$\frac{\partial u_I}{\partial t} = Y(t)\left(\Omega(t)\, u + \frac{\partial u}{\partial t}\right) + \dot{y}(t)$$

where $\Omega(t)$ is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

of angular velocities ω_i about the corresponding axes of the body frame.

The kinetic energy of the body is then given by

$$T = \tfrac{1}{2} \int_{\mathscr{B}} \| Y(\Omega u + u_t) + \dot{y} \|^2 \, dm,$$

where dm is the mass distribution on the body manifold $\mathscr{B}$. This formulation is rather general, as the form of the kinetic energy functional is independent of the fine structural details of the body. We shall only treat viscoelastic bodies wherein the material characteristics related to stress, strain, and material damping are modeled in terms of a potential energy function

$$V = V(Y, y, u)$$

and a dissipation function

$$\mathscr{D} = \mathscr{D}(u_t).$$

It is assumed that V is independent of velocity terms $(\dot{Y}, \dot{y}, u_t)$, and the dissipation function is defined as in [1]. (See also [6].)

Internal constraints on body deformations may be expressed by imposing suitable restrictions on the function space C. For example, we say the body is *partially rigid* if there is an $\mathbb{R}^3$ projection matrix $P : \mathbb{R}^3 \to \mathbb{R}^3$ such that for all $u \in C$

$$\| P(u(z_1) - u(z_2)) \| = \| P(z_1 - z_2) \|$$

for all $z_1, z_2 \in \mathscr{B}$. Other types of material constraints are discussed in TRUESDELL & NOLL [5], especially pp. 71–72, as well as in Section 3 below. In the present section, Example 2.1 will show that partial rigidity of the body gives rise to a set of steady-state rotations which is dramatically different from the case in which there are no internal material constraints. To provide the technical background for this example, we recall the following theorem on rotating elastic bodies.

Theorem 2.1 (BAILLIEUL & LEVI, 1987). *Supppose that C is a set of diffeomorphisms mapping $\mathscr{B}$ onto subsets of $\mathbb{R}^3$ such that the identity function, $u(z) \equiv z$, is in the interior of C in the natural topology inherited from the compact-open topology. In the absence of external forcing, the equations of motion of any system whose configuration space is $SO(3) \times \mathbb{R}^3 \times C$ as above are given by*

$$\dot{a} + \omega \times a + \int_{\mathscr{B}} u(z,t) \times Y^{-1} \ddot{y}\, dm = -S((Y^{-1} V_Y)_a), \tag{1}$$

$$u_{tt} + \omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t + \mathscr{D}_{\dot{u}} + V_u + Y^{-1} \ddot{y} = 0, \tag{2}$$

$$\int_{\mathscr{B}} [Y(u_{tt} + \omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t) + \ddot{y}]\, dm + V_y = 0, \tag{3}$$

where $a(\cdot)$ is the angular momentum defined by

$$a(t) = \int_{\mathscr{B}} u \times (u_t + \omega \times u)\, dm,$$

and for any 3×3 matrix M, $M_a = \frac{1}{2}(M - M^T)$ is the skew-symmetric part of M, and S maps the set of 3×3 skew-symmetric matrices to $\mathbb{R}^3$ according to the formula

$$S\left(\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

V_u and $\mathscr{D}_{\dot{u}}$ are Fréchet derivatives of V and $\mathscr{D}$ respectively.

Remark 2.1. If the hypothesis on the function space C is changed to admit internal material constraints, the variational argument leading to equation (2) (the equation describing the material configuration in the body frame) may have to be changed, and indeed the form of the equation may change. We shall illustrate this point in Example 2.1 below. Equation (1), on the other hand, is valid under a wide range of assumptions on the possible relative motions of material within the body. We note, for instance, that if the body is rigid and the origin of the body frame is chosen to coincide with the body center of mass, and if the there is no rotational potential $(V_Y \equiv 0)$, then the system (1) reduces to Euler's rigid body equations.

Much recent research on the dynamics and control theory of rotating structures has been aimed at characterizing solutions to this system of equations, and substantial progress has been made in certain special cases. (See, for example, [7].) While a complete theory of Lyapunov stability does not exist for continuum models of the form given in Theorem 2.1, we may nevertheless characterize steady state dynamics (in which the configuration function u has no time dependence) as follows.

Theorem 2.2. *Suppose a system with equations of motion* (1)–(3) *is undergoing a motion in which the configuration* $u(z, t) = u(z)$ *has no time dependence. Then*
(i) *the angular velocity* ω *is a constant* ω_∞,
(ii) *the angular momentum is a constant* $a_\infty = I_\infty \omega_\infty$, *where* I_∞ *is the corresponding steady state (constant) inertia tensor, and*
(iii) *the equilibrium rotations are aligned with a principal axis of the equilibrium inertia tensor, so that* $I_\infty \omega_\infty = \lambda \omega_\infty$, *where* λ *is an eigenvalue of* I_∞.

Proof. Because u has no dependence on time, equation (2) is rewritten

$$\omega \times (\omega \times u) + \dot{\omega} \times u + V_u + Y^{-1} \ddot{y} = 0.$$

Assume (without loss of generality) that the body $\mathscr{B}$ consists of points $(z_1, z_2, z_3)^T \in \mathbb{R}^3$, and further assume (also without loss of generality) that u explicitly depends on the coordinate variable z_3. Differentiating the above equation with respect to this variable we obtain

$$\omega \times (\omega \times u_{z_3}) + \dot{\omega} \times u_{z_3} + \frac{\partial}{\partial z_3} V_u = 0. \tag{4}$$

The dependence of terms in this equation on the variables z_j will play no further role in our argument, and to simplify notation we shall omit the z_3 subscripts and rewrite the equation as

$$\omega \times (\omega \times u) + \dot{\omega} \times u + \bar{c} = 0. \tag{5}$$

(Thus the variable u in equation (5) denotes the same quantity as the variable u_{z_3} in (4), and $\bar{c} = \frac{\partial}{\partial z_3} V_u$.) There is a 3×3 skew symmetric matrix U such that

$$U\dot{\omega} = \dot{\omega} \times u \quad \text{and} \quad U\omega = \omega \times u.$$

In terms of this matrix we may rewrite (5) as

$$\omega \times (U\omega) + U\dot{\omega} + \bar{c} = 0. \tag{6}$$

Given the skew-symmetric matrix U, there is an orthogonal matrix A such that

$$AUA^T = \begin{pmatrix} 0 & -\bar{u} & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We next define a new vector of angular velocities,

$$\bar{\omega} = A\omega,$$

which satisfies the following differential equation (coming from (6)):

$$A^T\bar{\omega}\times(UA^T\bar{\omega}) + UA^T\dot{\bar{\omega}} + \bar{c} = 0. \tag{7}$$

It can further be shown that multiplying both sides of this equation by A yields

$$\bar{\omega}\times(AUA^T)\,\bar{\omega} + AUA^T\dot{\bar{\omega}} + A\bar{c} = 0,$$

and because of the choice of A, this may be rewritten as

$$\begin{pmatrix} \dot{\omega}_1 - \omega_2\omega_3 \\ \dot{\omega}_2 + \omega_1\omega_3 \\ \omega_1^2 + \omega_2^2 \end{pmatrix} = \begin{pmatrix} c_2 \\ -c_1 \\ c_3 \end{pmatrix} \tag{8}$$

where $-c_j =$ the j-th component in the vector $\frac{1}{\bar{u}} A\bar{c}$, and where we have omitted the overbar on the ω_j's.

Multiplying the first two components by ω_1 and ω_2 respectively and adding, we obtain

$$c_2\omega_1 - c_1\omega_2 = \dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2. \tag{9}$$

From the last component of (8) it follows that the right-hand side is zero, and then (9) together with the last component of (8) implies ω_1 and ω_2 are constant, provided that c_1 and c_2 are not both zero.

To establish (i) in the case that $c_1 = c_2 = 0$, we show that there is no possible choice of body coordinates for any rotating rigid body in which the relationships $\dot{\omega}_1 = \omega_2\omega_3$ and $\dot{\omega}_2 = -\omega_1\omega_3$ hold, *unless* all ω_i's are constant. We proceed by noting there is an orthogonal change of basis $x = B\omega$ $(B \in SO(3))$ such that the angular velocity vector x is expressed in a principal axis coordinate system (cf. [8], p. 337) with components satisfying the system of differential equations

$$\dot{x}_i = \alpha_i x_j x_k$$

where $j = \sigma(i)$, $k = \sigma^2(i)$, $i = 1, 2, 3$, and σ is any permutation on the three symbols $i = 1, 2, 3$. As noted in [9] (p. 894), the definition of the coefficients α_i in terms of the principal moments of inertia implies that $|\alpha_i| < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. The form of the equations $\dot{\omega}_1 = \omega_2\omega_3$, $\dot{\omega} = -\omega_1\omega_3$ imposes (polynomial) conditions on the α_i's and the entries of the orthogonal matrix B. A straightforward analysis shows that if C is any orthogonal matrix and if the transformed vector $y = C^T x$ has its first two entries obeying equations of the form $\dot{y}_1 = \beta_1 y_2 y_3$, $\dot{y}_2 = \beta_2 y_1 y_3$, then in fact C must be a permutation matrix (a nonsingular matrix in which each column has a single 1 and all other entries equal 0). The β_i's therefore must be equal, modulo a permutation, to the α_i's and in, particular we must have $|\beta_i| < 1$, which is not consistent with the form of the equations for ω_1, ω_2 unless all ω_i's are constant. This completes the proof of statement (i) in the theorem.

Statement (ii) is an immediate consequence of (i), and (iii) follows since ω_∞ must satisfy $\omega_\infty \times I_\infty \omega_\infty = 0$. $\square$

Remark 2.2. The key ideas in proving this result are drawn directly from Section 5 of our previous paper [1], where a similar result was given for a special rotating body-beam system. A similar result is central to the work reported more recently in [10].

A more detailed understanding of steady-state rotations requires a more precise specification of the structural models to be investigated. In [1], a very simple anisotropic rod model was used to investigate the relationship between equilibrium rod shapes $u(z)$ and angular velocities ω_∞. In the next section, we shall examine models of thin beams undergoing planar rotation, including several "geometrically exact" models. Beam models are of course idealizations of three-dimensional continua, but such models are frequently further simplified by both kinematic assumptions (e.g., it is assumed that beam elongation is negligible) or analytic assumptions (e.g., only first-order terms are retained in the expression for strain as is the case in the Euler-Bernoulli model). Our purpose in examining such questions will be to understand the extent to which they affect the observed qualitative dynamics. In the context of the beam models of the next section, we find standard simplifying assumptions invariably imply stiffness or rigidity, and in general such rigidity seems to have a significant effect on qualitative dynamics.

Already, in terms of Theorem 2.2, we can show that the hypothesis that there are no internal material constraints is crucial for the conclusions of the theorem. To see what is obtained by relaxing this assumption, the following example is instructive.

Example 2.1. ([11]). Consider a mechanical system consisting of a rigid body with a one-degree-of-freedom mass-spring attachment. Here we denote the rigid body tensor of inertia by I, and orient the coordinate axes x, y, z along the eigendirections of I, denoting the corresponding eigenvalues, i.e., the moments of inertia, by $I_1 > I_2 \geqq I_3$. Let a point mass m be attached to a linear spring which is attached to the rigid body with the mass-spring assemblage constraining the motion of the mass to the body x-axis. Let x_0 denote the equilibrium position (when the system is at rest), and assume the center of mass of the rigid body to be fixed in space — or alternatively, take a mass $m_1 = m$ positioned symmetrically at $-x$, with symmetric neutral configuration and initial conditions. Let $k(x - x_0)$ be the restoring force of the spring. The equations of motion are

$$\begin{aligned} I\dot{\omega} + \omega \times I\omega + 2x\dot{x}\omega^1 + x^2\dot{\omega}^1 + x^2\omega_1(0, -\omega_3, \omega_2)^T = 0,& \\ \ddot{x} + c\dot{x} + k(x - x_0) = (\omega_2^2 + \omega_3^2)& \end{aligned} \tag{10}$$

where $\omega^1 = (0, \omega_2, \omega_3)^T$, and where $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity expressed in the body frame. $c\dot{x}$ is the frictional force acting upon the mass, and we have assumed without loss of generality that $m = 1$.

The total energy of the system is given by

$$E = \tfrac{1}{2}\,\omega^T I\omega + \tfrac{1}{2}\,[\dot{x}^2 + (\omega_2^2 + \omega_3^2)\,x^2] + \tfrac{1}{2}\,k(x - x_0)^2,$$

with the expression in brackets giving the kinetic energy of the mass. One easily checks the dissipation relation

$$\dot{E} = -c\dot{x}^2.$$

The angular momentum of the system (in the body frame) is given by $M = I(x)\,\omega$, where $I(x) = \operatorname{diag}(I_1, I_2 + x^2, I_3 + x^2)$, and from the left-invariance of the Lagrangian it follows that M is a conserved quantity.

Just as in Theorem 2.2 above, it follows that if $I_2 \neq I_3$, then the body angular velocity $\omega(t) \to \omega_\infty = \text{const}$. (See [11], Theorem 3.2.) It is important to note that this conclusion is false in general, however, since if $I_2 = I_3$, we find on writing out the components of equations of motion (10) as

$$\dot{\omega}_1 = 0,$$

$$\dot{\omega}_2 = \frac{I_2 - I_1 + x^2}{I_2(1 + x^2)}\,\omega_1\omega_3 - \frac{2x\dot{x}\omega_2}{I_2(1 + x^2)},$$

$$\dot{\omega}_3 = \frac{I_1 - I_2 - x^2}{I_2(1 + x^2)}\,\omega_1\omega_2 - \frac{2x\dot{x}\omega_3}{I_2(1 + x^2)},$$

$$\ddot{x} + c\dot{x} + k(x - x_0) = \omega_2^2 + \omega_3^2,$$

a "typical" solution tends to $x = \text{const}$, $\omega(t) + (\omega_1, a\cos b(t - t_0), a\sin b(t - t_0))$ with constant a, b, and ω_1. Physically, this says that even in the pressence of elasticity some structures need not tend to a pure rotation but rather may asymptotically tend to a precessing motion. This phenomenon arises precisely because the relative motion of the mass and body are constrained (to lie along the body x-axis). In the next section, we provide further examples of ways in which the qualitative dynamics of rotating systems depend crucially on constraints that exist on relative motions of system components.

3. Beam Theories: A Hierarchy of Geometrically Exact Models

Following early work reported in [12], a number of authors have investigated the rotational mechanics of a rigid body with an elastic beam attachment. While these early papers and even work as recent as [3] incorporate fairly naive models of beam dynamics, more recent work has concentrated on so-called *geometrically exact* formulations. By the term "geometrically exact" we mean that in developing our beam model we do not make any *ad hoc* assumptions leading to the neglect of kinematic terms regarded as small. (Cf. ANTMAN [13].)

While any beam model provides a highly idealized description of a three-dimensional solid, it is possible to isolate certain assumptions underlying various models which have a significant bearing on predicted physical behavior. To amplify this remark, we shall examine a hierarchy of models of a rotating mechanical complex consisting of an elastic beam clamped to a rigid body. In order to keep issues involving the qualitative dynamics sharply in focus, we shall restrict our investigation to planar systems.

Pursuing a so-called director theory, we develop our model as a set of equations having the parameter of a certain curve and time as the only independent variables. We shall assume that when the rod is in its neutral position, it is described by the curve

$$\left\{\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 : \ 0 \leqq x \leqq l\right\}.$$

Thus the beam is parametrized by x with $0 \leqq x \leqq l$. We associate to each point x three quantities $u_1(x, t)$, $u_2(x, t)$ and $\alpha(x, t)$. $u_1(x, t)$ and $u_2(x, t)$ denote respective deformations or displacements of the material point x in the x and y directions at time t. $\alpha(x, t)$ denotes the "orientation of the material" at x at time t. These quantities are illustrated in Figure 1. Throughout this section we shall assume the beam is clamped to the rigid body at its left-hand point so as to produce the boundary conditions $u_1(0, t) = u_2(0, t) = \alpha(0, t) = 0$ while the right end point is free, leading to the boundary conditions $u_{1x}(l, t) = u_{2x}(l, t) = \alpha_x(l, t) = 0$. The role of these functions in characterizing the deformations is illustrated by the examples depicted in Figure 2.

The strains associated with this formulation are the vector components of

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \Gamma \end{pmatrix} = \begin{pmatrix} \cos\alpha(x) & \sin\alpha(x) & 0 \\ -\sin\alpha(x) & \cos\alpha(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + u_{1x} \\ u_{2x} \\ \alpha_x \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

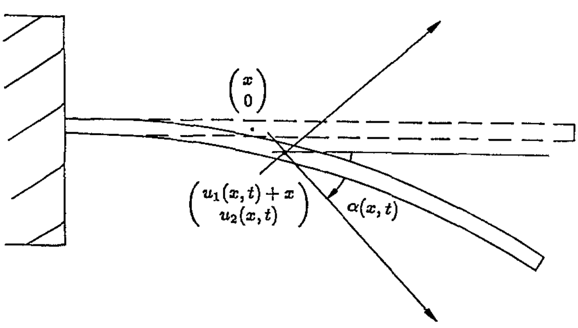

Fig. 1. The parametrization of a beam in the plane.

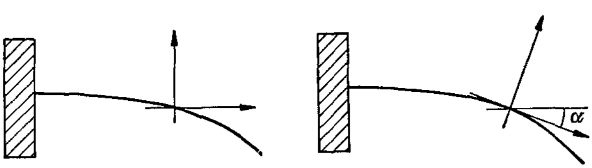

Fig. 2. Deformation of the rod is principally due to bending when $\tan\alpha(x) \equiv \dfrac{u_{2x}}{1 + u_{1x}}$. The left-hand figure displays considerably more pronounced shear deformation than the right-hand figure.

(These are the deformations per unit length in each of the coordinate directions.) Our model assumes a linear Hooke's law relating strains to stresses via

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau\end{pmatrix} = \begin{pmatrix}\mu_1\varepsilon_1\\ \mu_2\varepsilon_2\\ k\Gamma\end{pmatrix},$$

with a corresponding elastic potential given by

$$V = \tfrac{1}{2}\int_0^l (\mu_1\varepsilon_1^2 + \mu_2\varepsilon_2^2 + k\Gamma^2)\,dx.$$

Next consider a planar beam characterized by such an elastic potential and which is also attached to a planar rigid body as depicted in Figure 3. We suppose that this system is free to move in the plane. (That is, there are no constraining forces impeding translational or rotational motions.) Following the modeling approach of [1], we take the configuration space to be $SO(2)\times\mathbb{R}^2\times C$ where C is the space of $\mathbb{R}^3$-valued functions (u_1, u_2, α) described above, which account for displacements of the rod with respect to the "body frame." The remaining variables describing the kinematics are $\theta = \theta(t)$ and the pair $(y_1(t), y_2(t))$ giving respectively the orientation and location of the origin of the body frame with respect to the designated inertial frame. We shall assume all these quantities are at least twice-differentiable functions of their arguments. The basic kinematic equation describing the location and orientation of each infinitesimal rod segment $[x, x+dx]$ with respect to the inertial frame is

$$\begin{pmatrix}u_{1I}(x,t)\\ u_{2I}(x,t)\\ \alpha_I(x,t)\end{pmatrix} = \begin{pmatrix}\cos\theta(t) & -\sin\theta(t) & 0\\ \sin\theta(t) & \cos\theta(t) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}u_1(x,t)+x\\ u_2(x,t)\\ \alpha(x,t)\end{pmatrix} + \begin{pmatrix}y_1(t)\\ y_2(t)\\ \theta(t)\end{pmatrix}.$$

From this, we may deduce the form of the kinetic energy:

$$T = \tfrac{1}{2}I_b\dot\theta^2 + \tfrac{1}{2}\int_0^l I(\alpha_t + \dot\theta)^2\,dx + m_b\langle y, YJ\dot\theta c\rangle$$
$$+\tfrac{1}{2}m_b\|\dot y\|^2 + \tfrac{1}{2}\int_0^l \varrho\,\|Y Du + \dot y\|^2\,dx$$

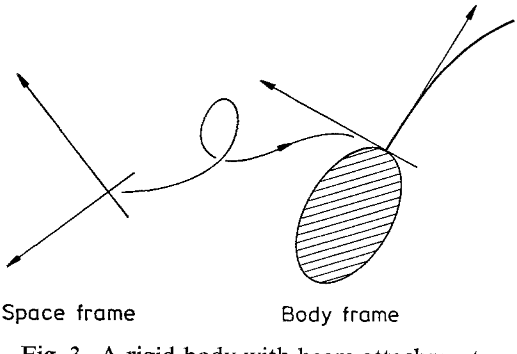

Fig. 3. A rigid body with beam attachment.

where $\langle\cdot,\cdot\rangle$ is the standard $\mathbb{R}^2$ inner product, and where we have introduced the notation

$$u=\begin{pmatrix}u_1+x\\ u_2\end{pmatrix},\quad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix},\quad Y=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},$$

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

I_b = moment of inertia of the rigid body,

m_b = mass of the rigid body,

I = moment of inertia of a cross-sectional area of the (uniform) rod,

ϱ = constant mass density of the rod,

$c=\begin{pmatrix}c_1\\ c_2\end{pmatrix}$ = body-frame center of mass of the rigid body,

$D=\frac{d}{dt}+J\dot\theta$ = the *rotational derivative* operator,

$$a(t)=(I_b+Il)\,\dot\theta+\int_0^l[\varrho u^TJ^T\,Du+I\alpha_t(x,t)]\,dx$$

is the total angular momentum,

$\frac{\delta V}{\delta u_1},\ldots$ are the Fréchet derivatives of the potential energy functional V.

From the Lagrangian $T-V$ a variational argument shows that the equations of motion are given by

$$\dot a(t)+\left[m_bc+\int_0^l\varrho u\,dx\right]^TJ^TY^T\ddot y=0, \tag{11}$$

$$Y^T\ddot y+\frac{1}{m_b+\varrho l}D^2\left[m_bc+\int_0^l\varrho u\,dx\right]=0, \tag{12}$$

$$\varrho[D^2u+Y^T\ddot y]+\frac{\delta V}{\delta u}=0, \tag{13}$$

$$I(\alpha_{tt}+\ddot\theta)+\frac{\delta V}{\delta\alpha}=0. \tag{14}$$

Here $\frac{\delta V}{\delta u}=\left(\frac{\delta V}{\delta u_1},\frac{\delta V}{\delta u_2}\right)^T$. In this form, the equations are geometrically exact (in the sense discussed above), but they are also rather complex. Our purpose in the remainder of this section is to discuss a number of simplifying assumptions and to examine the consequences of these assumptions in terms of the resulting qualitative dynamics.

3.1. Zero-strain or infinite-stiffness assumptions

Depending on the kinematics of the system in question, one may be justified in seeking to simplify the above model by assuming one or more of the strains to be zero. If one of ε_i or Γ is identically zero, then one of the kinematic variables u_i, α may be eliminated from (11)–(14). To illustrate, we assume that shear deformations may be neglected by setting $\varepsilon_2 \equiv 0$, and express α in terms of u_1, u_2 by writing

$$\begin{pmatrix} \cos\alpha(x) \\ \sin\alpha(x) \end{pmatrix} = \begin{pmatrix} \dfrac{1+u_{1x}}{\sqrt{(1+u_{1x})^2+u_{2x}^2}} \\ \dfrac{u_{2x}}{\sqrt{(1+u_{1x})^2+u_{2x}^2}} \end{pmatrix}.$$

The strain energy then reduces to

$$V = \tfrac{1}{2}\int_0^l (\mu_1\varepsilon_1^2 + k\Gamma^2)\,dx$$

where

$$\varepsilon_1 = \sqrt{(1+u_{1x})^2+u_{2x}^2} - 1,$$

$$\Gamma = \frac{u_{2xx}(1+u_{1x}) - u_{2x}u_{1xx}}{(1+u_{1x})^2+u_{2x}^2}.$$

A further assumption, which is frequently made, is that we may ignore terms which are of high order and which may be deemed to be "small" in a given application. An argument for this approach in the present context would exploit an assumption of small deformations of the rod to justify omitting certain terms of quadratic and higher order in the deformation variables u_1 and u_2. Restricting our attention to potential energy terms, we expand ε_1 and Γ to obtain

$$\varepsilon_1 = u_{1x} + \text{higher order terms}, \tag{15}$$

$$\Gamma = u_{2xx} + \text{higher order terms}. \tag{16}$$

Ignoring all but the first-order terms gives a particularly simple form for $V = \int_0^l (\mu_1 u_{1x}^2 + k u_{2xx}^2)\,dx$, and it is easy to write down the Fréchet derivative terms (evaluated at $u = 0$):

$$\begin{pmatrix} \dfrac{\delta V}{\delta u_1} \\ \dfrac{\delta V}{\delta u_2} \end{pmatrix} = \begin{pmatrix} -u_{1xx} \\ u_{2xxxx} \end{pmatrix}.$$

Putting these expressions into equation (13) gives a two-dimensional version of the rotating Euler-Bernoulli beam model that was described briefly in [1] (Section 4). This model enjoys many features common to any rotating structure containing elastic components. There are important differences, however, between the no-shear assumption made above and the subsequent assumption that terms of higher order can be ignored. The first assumption is easily interpreted in terms of the material properties and kinematics of the beam. Interpreting the second assumption is somewhat more difficult, and as we shall describe later, unexpected dynamical consequences may be manifested in the models (even when the systems are in a dynamic steady state).

3.2. *Incorporating further stiffness assumptions*

The relative magnitude of the strains in the kinematics of beam deformations depends on the constitutive properties of the beam material. In the analysis of thin rods made from isotropic material (e.g., mild steel), longitudinal and shear deformations are far less significant than deformations due to bending. Hence, it is reasonable to impose the constraints $\varepsilon_1 = \varepsilon_2 \equiv 0$. Together these imply that we may express $u_1(x,t)$, $u_2(x,t)$ in terms of $\alpha(x,t)$:

$$u(x,t) = \begin{pmatrix} u_1(x,t) + x \\ u_2(x,t) \end{pmatrix} = \int_0^x \begin{pmatrix} \cos\alpha(s,t) \\ \sin\alpha(s,t) \end{pmatrix} ds$$

$$= \int_0^x e^{i\alpha(s,t)}\, ds. \tag{17}$$

Under this assumption, motions of the beam are subject to forces arising from an elastic potential

$$V = \tfrac{1}{2} \int_0^l \varkappa \Gamma^2\, dx.$$

In this case, the rotating-structure dynamics involve the variables x, θ, and α and may be expressed as follows:

Theorem 3.1. *In the absence of external forcing, the equations of motion for an inextensible, unshearable beam* $(\varepsilon_1 \equiv 0, \varepsilon_2 \equiv 0)$ *attached to a planar rigid body as described above are*

$$\dot{a}(t) + Y^{-1}\ddot{y} \times_2 \left(m_b c + \int_0^l \varrho u(x,t)\, dx \right) = 0, \tag{18}$$

$$Y^{-1}\ddot{y} + \frac{1}{m_b + \varrho l} D^2 \left[m_b c + \int_0^l \varrho u(x,t)\, dx \right] = 0, \tag{19}$$

$$\int_x^l \varrho \left(\int_0^s \cos(\alpha(x,t) - \alpha(\sigma,t))\, (\alpha_{tt}(\sigma,t) + \ddot{\theta}) \right.$$

$$\left. + \sin(\alpha(x,t) - \alpha(\sigma,t))\, (\alpha_t(\sigma,t) + \dot{\theta})^2\, d\sigma \right) + (ie^{i\alpha(x,t)})^T\, Y^{-1}\ddot{y})\, ds$$

$$+ \varrho I I(\alpha_{tt} + \ddot{\theta}) - \varkappa \alpha_{xx}(x,t) = 0. \tag{20}$$

Proof. From the relation (17), we may write the Lagrangian $L = T - V$ as a function of α together with its derivatives. Equations (18) and (19) are the Euler-Lagrange equations with respect to the variables $\theta(t)$ and $(y_1(t), y_2(t))$ respectively. Equation (20) is the Euler-Lagrange equation with respect to the function space variable $\alpha(\cdot, t)$. □

Corollary 3.1. *In steady-state,* $\dot{\theta} = \omega = \text{const.}$ *and* α *does not depend on* t, *so that* (20) *takes the form*

$$\varkappa\alpha''(x) + \begin{pmatrix}\cos\alpha(x)\\ \sin\alpha(x)\end{pmatrix} \times_2 \int_x^l \varrho\left(\int_0^s \begin{pmatrix}\cos\alpha(\sigma)\\ \sin\alpha(\sigma)\end{pmatrix} d\sigma - \begin{pmatrix}\bar{c}_1\\ \bar{c}_2\end{pmatrix}\right) ds\,\omega^2 = 0 \tag{21}$$

where $\alpha''(x) = \dfrac{d^2\alpha}{dx^2}$, *the steady-state body-beam center of mass is denoted by*

$$\begin{pmatrix}\bar{c}_1\\ \bar{c}_2\end{pmatrix} = \frac{1}{m_b + \varrho l}\left[m_b c + \int_0^l \varrho \int_0^s \begin{pmatrix}\cos\alpha(\sigma)\\ \sin\alpha(\sigma)\end{pmatrix} d\sigma\, ds\right],$$

and where the planar cross-product $\times_2$ *is defined by*

$$\begin{pmatrix}a\\ b\end{pmatrix} \times_2 \begin{pmatrix}c\\ d\end{pmatrix} = ad - bc.$$

Equation (21) *has a first integral*

$$\int_x^l \varrho \int_x^s \begin{pmatrix}\cos\alpha(\sigma)\\ \sin\alpha(\sigma)\end{pmatrix} d\sigma\, ds \times_2 \left[\int_0^x \begin{pmatrix}\cos\alpha(\sigma)\\ \sin\alpha(\sigma)\end{pmatrix} d\sigma - \begin{pmatrix}\bar{c}_1\\ \bar{c}_2\end{pmatrix}\right]\omega^2 - \varkappa\alpha'(x) = 0. \tag{22}$$

Proof. Equation (19) expresses the fact that linear momentum is conserved. The Marsden-Weinstein reduction of the phase space ([8], p. 298) may be explicitly carried out by using (19) to eliminate $\ddot{y}$ from (20). Equation (21) is just the standard equilibrium equation of the reduced system. Differentiation of (22) easily yields (21). □

Remark 3.1. These equations may be interpreted as prescribing balance laws between centrifugal and elastic forces acting on our system. We shall refer to the dynamic equilibrium configurations which are defined as the *rotational elastica.* (Cf. Love, [14], Ch. 19.) While a complete classification of solutions is beyond the scope of the present paper, such a classificaton is nevertheless important for applications including the development of a control theory for rotating structures. In Section 4, we shall discuss linearizations about these dynamical equilibria.

Before delving further into the details of this model, it will be useful to briefly note the qualitative features which arise under certain alternative assumptions on the strains. The assumption of no shear ($\varepsilon_2 = 0$) and no bending ($\Gamma = 0$) leads to purely longitudinal deformations. On the other hand, if $\varepsilon_1 = 0$ and $\Gamma = 0$, then the relative displacement of each particle is to move along a straight line perpendicular to the body-frame x-axis. The resulting equations of motion are given as follows.

Theorem 3.2. (i) *Under the constraints* $\varepsilon_2 = \Gamma \equiv 0$, *both* α *and* u_2 *are identically zero, and* u_1 *satisfies the partial differential equation*

$$\varrho u_{tt} = \varrho\dot{\theta}^2(u + x) + \varrho(\dot{v}_1 + v_2\dot{\theta}) + \mu_1 u_{xx} \tag{23}$$

subject to the boundary conditions $u(0, t) = u_x(l, t) = 0$. *Here*

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}$$

is the velocity of the body-beam attachment point in the body frame.
(ii) *Under the constraints* $\varepsilon_1 = \Gamma \equiv 0$, *both* α *and* u_1 *are identically equal to zero, and* u_2 *satisfies the partial differential equation*

$$\varrho u_{tt} = \varrho\dot{\theta}^2 u - \varrho(x\ddot{\theta} + \dot{\theta}v_1 + \dot{v}_2) + \mu_2 u_{xx} \tag{24}$$

subject to the boundary conditions $u(0, t) = u_x(l, t) = 0$, *with* v_1 *and* v_2 *as in case* (i).

Proof. In both cases the vanishing of α and u_i follows from the definitions of the strains Γ, ε_i together with the boundary conditions. With the Lagrangian $T - V$ expressed in terms of $\dot{\theta}$, $\dot{y}$, and the nonzero kinematic variable $u_j(x, t)$, a straightforward variational argument leads to the equations (23) and (24) in the respective cases. $\square$

In studying solutions to (11)–(14) and the various constrained counterparts (20), (23), and (24), we shall be especially interested in relative equilibrium solutions in which the configuration variables u_1, u_2, and α have no dependence on time. Such relative equilibria are in part characterized by the following:

Lemma 3.1. *In either case treated in Theorem 3.2, if* u *has no dependence on time, then* $\dot{\theta} = \omega \equiv$ const. *If, moreover, the center of mass of the body-beam system*

$$\bar{c} = \frac{1}{m_b + \varrho l}\left[m_b c + \int_0^l \varrho u\, dx\right]$$

is at rest with respect to the inertial frame, then

$$v(t) \equiv -\omega J\bar{c}.$$

Proof. (i) If u does not depend on t, then differentiating both sides of (23) with respect to t yields the equation

$$2\varrho\dot{\theta}\ddot{\theta}(u(x) + x) = -\varrho\frac{d}{dt}(\dot{v}_1 + v_2\dot{\theta}).$$

We assert the product $\dot{\theta}\ddot{\theta}$ must be identically zero. To prove this, note that the right-hand side is independent of x, and either $\dot{\theta}\ddot{\theta} = 0$ or $u(x) + x = \lambda$ where λ is constant with respect to x. But the boundary conditions prevent $u(x) + x$

from being constant, and hence $\dot{\theta}\ddot{\theta} \equiv 0$ as asserted. Note that this product can be zero only if $\ddot{\theta} \equiv 0$, and thus we have proved that $\dot{\theta}$ is constant.

We may now integrate (19) twice to obtain

$$y + Y\bar{c} = at + b$$

where a and b are constants of integration. Note that the quantity on the left-hand side of this equation is just the space-frame position of the body-beam center of mass with respect to the inertial frame. Under the hypothesis that the body-beam center of mass undergoes no motion with respect to the space frame, we have $a = 0$, and the result $v_1(t) \equiv \omega\bar{c}_2, v_2(t) \equiv -\omega\bar{c}_1$ follows easily.

This proves the proposition in case (i). The proof of case (ii) is similar and is omitted. □

An immediate consequence of these results is the following characterization of relative equilibrium configurations.

Theorem 3.3. *Suppose that in the body frame the rigid-body center of mass lies on the line defined by the beam in its neutral position, i.e.,*

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}.$$

(i) *Let the constraints $\varepsilon_2 = \Gamma \equiv 0$ hold (so that $\alpha = u_2 \equiv 0$). For the relative equilibrium dynamics of the body-beam system in which there is no t-dependence in $u_1(x, t) = u_1(x)$, the rotation rate $\dot{\theta}$ is some constant ω, and $u_1(\cdot)$ satisfies the two-point boundary value problem*

$$\mu_1 u_1''(x) + \varrho\omega^2(u_1(x) + x - c_1) = 0, \tag{25}$$

$$u_1(0) = u_1'(l) = 0. \tag{26}$$

(ii) *Let the constraints $\varepsilon_1 = \Gamma \equiv 0$ hold (so that $\alpha = u_1 \equiv 0$). For the relative equilibrium dynamics of the body-beam system in which there is no t-dependence in $u_2(x, t) = u_2(x)$, the rotation rate rate $\dot{\theta}$ is some constant ω and $u_2(\cdot)$ satisfies the two-point boundary value problem*

$$\mu_2 u_2''(x) + \varrho\omega^2 u_2(x) = 0, \tag{27}$$

$$u_2(0) = u_2'(l) = 0. \tag{28}$$

Remark 3.2. It is perhaps useful to refer the foregoing discussion to Figure 4. Here we have arranged the constitutive assumptions under discussion into a hierarchy. Treating those along the third row, we have already remarked that the assumptions of inextensibility and no shear ($\varepsilon_1 = \varepsilon_2 \equiv 0$) seem quite reasonable for slender uniform beams. On the other hand, the combined assumptions of inextensibility (in the direction of α) and no bending ($\varepsilon_1 = \Gamma \equiv 0$) lead to a model which, following the terminology of KRISHNAPRASAD & MARSDEN, we may call a *shear beam*. The dynamics of the rotating shear beam were studied extensively in [3].

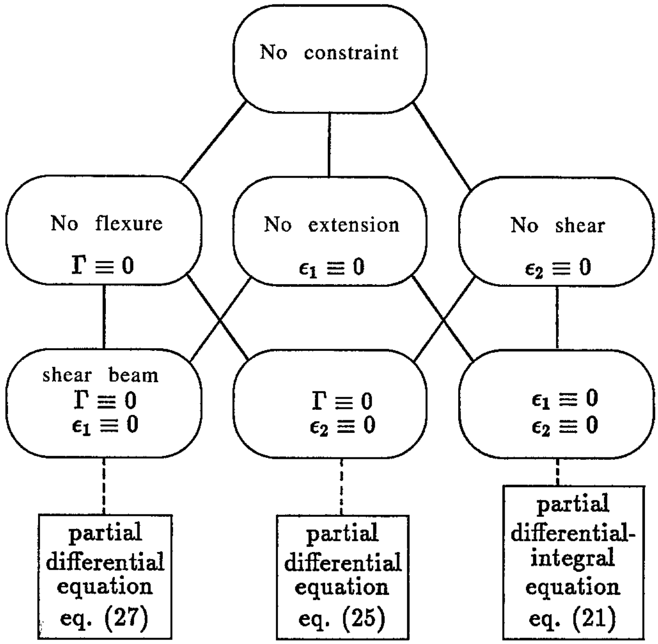

Fig. 4. A hierarchy of kinematic assumptions in modeling the configuration dynamics of the thin beam: The square boxes list the model type under each set of assumptions.

Remark 3.3. Each of the models given by (20), (23), and (24) manifests the assumed rigidity in a different way. In both (23) and (24), each particle of the beam can move only in a straight line with respect to the body coordinate frame. (In (23) the particles in the continuum are constrained to horizontal motions, while in (24) the particles move vertically.) In (20) rigidity manifests itself in the model being spatially nonlocal in that the equation includes terms which are integrated over the length of the beam with respect to the spatial variable.

Remark 3.4. Interesting qualitative differences may be observed in the relative equilibria determined by (21), (25) and (27). In all three cases, these equations together with the constraint prescribed by the conservation of angular momentum determine a set of possible constant rotation rates ω which can occur in steady state. Unlike the first two cases, however, the possible rotation rates $0 < \omega_1 < \omega_2 < \dots$ determined by the homogeneous boundary value problem (27) are *independent* of the angular momentum of the system. A finite-dimensional analog of this distinction is illustrated by the planar systems depicted in Figure 5. Here we consider two planar rigid bodies consisting of uniform discs of mass m and having moment of inertia I about the center of mass. As in the above body-beam models, we assume that the bodies here may move freely (translate and rotate) in the plane. A mass-spring system attached to body (a) permits a point mass m_1 to move along a line attached to the body so that it passes through the body center of mass. The neutral position of the mass-spring system when the body is not in motion with respect to the spatial reference frame is at a distance $r > 0$ from the body center of mass. It is important to distinguish this *inertial* neutral position from other similar configurations in which the point mass is stationary with respect to the rigid body but where the entire system is undergoing a steady constant rotation. Body (b) also has a mass-spring system with point mass m_1 attached, but in this case the motion is *perpendicular* to a line passing through the center of mass of the rigid body. When this system is not in motion with respect to the spatial reference frame, the (inertial) neutral position of the point mass is at the point where its line of motion relative to the disc perpendicularly intersects a line passing through the center of mass of the body. Again let r denote the distance of this neutral point from the center of mass of the body. Let k denote the spring constants in both systems (a) and (b).

The steady-state motions of the system depicted in Figure 5(a) require a balance between the spring and centrifugal forces as prescribed by the equation

$$(k - \alpha\omega^2)\, x - kr = 0 \tag{29}$$

where $\dot{\theta} = \omega$ is the constant rate of rotation, x is the distance of the point mass from the body center of mass, and $\alpha = mm_1/(m + m_1)$. While this equation exhibits a simple continuous relationship between angular rate ω and displacement x of the mass, it is important to note that for any given angular momentum

$$M = (I + \alpha x^2)\, \omega \tag{30}$$

there are, depending on M, either one or three pairs of values (ω, x) satisfying both (29) and (30). Nevertheless, as the parameter M varies continuously, so do the corresponding rotation rates ω.

The steady-state motions depicted in Figure 5(b) depend in a dramatically different way on the angular momentum parameter M. The position of the point mass will be displaced (vertically in the local frame) by an amount y from its inertial neutral position where y and ω satisfy the simultaneous equations

$$(k - \alpha\omega^2)\, y = 0,$$

$$[I + \alpha(r^2 + y^2)]\, \omega = M.$$

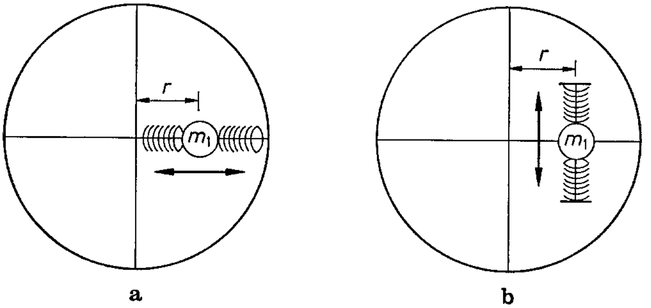

Fig. 5. Planar rigid bodies with mass-spring attachments.

In this case either $y = 0$ or $\omega^2 = k/\alpha$, independent of M. This is a precise finite-dimensional analog of continuum models such as the steady-state rotating shear beam dynamics prescribed by equation (27) and the beam model described by BLOCH in [19]. Stability analyses for the steady-state solutions of (27) may be carried out along the lines of [1], but we shall not pursue this here.

Remark 3.5. The inextensible no-shear model (20) may be viewed as the continuous limit of a spatially discrete system consisting of a planar kinematic chain with spring loaded joints. Details of this type of approximation appear in [16], and steady-state and dynamic analysis of rotating planar chain systems appear in [15, 17, 18].

4. Localization by Linearization

Our goal in this section is to show that the linearization of the integral-differential equation governing the inextensible rotating beam is a partial differential equation precisely when the linearization is performed at a relative equilibrium in which the beam is straight and aligned with the body-frame x-axis. A large body of literature has appeared in which the control of flexible structures is studied extensively in the context of models given by partial differential equations such as (23) and (24). Indeed, many authors (see, e g., [20], [21]) take partial differential equation models as their starting point in studying the control of structures. While both the ample theory and extensive literature on applications of partial-differential equations make such models attractive, the inextensible, shear-free beam model described in Theorem 3.1 suggests that qualitatively correct models of rotating structures may be expected to be both nonlinear and spatially nonlocal. (E.g., in the inextensible shear-free model, the dynamical equations unavoidably involve integrals with respect to the spatial variable.) This observation is supported by [15] where it is noted that the model (18)–(20) may be obtained as a limit of finite-difference approximations defined in terms of kinematic chains with spring-loaded joints. It is further observed in [15] (and also in [22]) that the classical beam models involving fourth-order partial differential equations arise as linearizations of the model (18)–(20) about certain relative equilibria. Part of the significance of these results is that numerical computations involving partial differential equations are simpler in the context of classical numerical methods than computations involving hybrid differential-integral equations. Indeed partial differential equation models are spatially localized in that difference approximations with respect to the spatial variable show the dynamics of each point in the continuum to be directly coupled only to its nearest few neighbors. Thus implementations of parallel algorithms for solution of such equations are easily carried out using only nearest neighbor interconnections. For the general nonlocal, nonlinear model (18)–(20), however, time derivatives of each element in the continuum depend on every other element. Since each point in the continuum interacts with every other point, the naive approach to parallel numerical integration of the equations of motion will involve interconnections between every pair of elements. Various approaches to simplifying this "massively parallel" interconnection pattern are suggested by the recent literature on the computational structure of the n-body problem (see, e.g., [23]), but here one has to carefully examine *a priori* assumptions which rule out certain nonlinear interactions among particles in the system.

A detailed discussion of computational issues will not be pursued here. The following theorem indicates that the partial differential equation models presented in [15] are essentially the only cases in which linearization of (18)–(20) about a relative equilibrium results in a partial differential equation for the beam configuration in the planar body-beam problem under discussion.

Theorem 4.1. *If the dynamics* (18)–(20) *are linearized about a relative equilibrium rotation characterized by* $\alpha(x,t)=\alpha_0(x)$ *and satisfying* (21) *and* $\dot\theta(t)\equiv\omega=\text{const.}$, *the linearized equation* (20) *for* $\alpha(x,t)=\alpha_0(x)+y(x,t)$ *may be written as a partial differential equation in the variational term* $y(x,t)$ *if* (i) $c_2=0$ *and* (ii) $\alpha_0(x)\equiv 0$. *When these conditions are satisfied, the partial differential equation for* y *is*

$$\delta\ddot\theta+\varrho y_{tt}-\varrho lI y_{ttxx}+\varkappa y_{xxxx}+\left[\varrho y+\frac{\partial}{\partial x}(g(x)\,y_x)+g'(x)\,y_x\right]\omega^2=0 \tag{31}$$

where

$$g(x)=\varrho\,\frac{l^2-x^2}{2}-\frac{1}{m_b+\varrho l}\left(m_b c_1+\frac{\varrho l^2}{2}\right)(l-x). \tag{32}$$

Proof. The Marsden-Weinstein reduction using the conservation of linear momentum equation (19) allows $\ddot y$ to be eliminated from (20) to give

$$\begin{aligned}\int_x^l\int_0^s F(x,\sigma,t)\,d\sigma-\frac{1}{m_b+\varrho l}\Bigg[&m_b(-ie^{i\alpha(x,t)})^T\cdot c\dot\theta^2\\&+m_b(e^{i\alpha(x,t)})^T\cdot c\ddot\theta+\int_0^l\int_0^\zeta F(x,\tau,t)\,d\tau\,d\zeta\Bigg]\,ds\\&+\varrho lI(\alpha_{tt}(x,t)+\ddot\theta)-\varkappa\alpha_{xx}(x,t)=0,\end{aligned} \tag{33}$$

where $F(x,\sigma,t)=\varrho[\cos(\alpha(x,t)-\alpha(\sigma,t))\,(\alpha_{tt}(\sigma,t)+\ddot\theta)+\sin(\alpha(x,t)-\alpha(\sigma,t))\,(\alpha_t(\sigma,t)+\dot\theta)^2]$. Substituting $\dot\theta(t)=\omega+\delta\dot\theta(t)$ and $\alpha(x,t)=\alpha_0(x)+y(x,t)$ and retaining terms of order 1 in $\delta\dot\theta$ and y, we obtain

$$\begin{aligned}\int_x^l\int_0^s \hat F(x,\sigma,t)\,d\sigma-\frac{1}{m_b+\varrho l}\Bigg[&m_b(e^{i\alpha_0(x)})^T\cdot c(\delta\ddot\theta+\omega^2 y(x,y))\\&+2m_b(ie^{i\alpha_0(x)})^T\cdot c\omega\,\delta\dot\theta+\int_0^l\int_0^\zeta \hat F(x,\tau,t)\,d\tau\,d\zeta\Bigg]\,ds\\&+\varrho lI(y_{tt}(x,t)+\delta\ddot\theta)-\varkappa y_{xx}(x,t)=0,\end{aligned} \tag{34}$$

where

$$\begin{aligned}\hat F(x,\sigma,t)=\varrho[&\cos(\alpha_0(x)-\alpha_0(\sigma))\,(y_{tt}(\sigma,t)+\delta\ddot\theta)\\&+\cos(\alpha_0(x)-\alpha_0(\sigma))\,(y(x,t)-y(\sigma,t))\,\omega^2\\&+2\sin(\alpha_0(x)-\alpha_0(\sigma))\,(y_t(\sigma,t)+\delta\dot\theta)\,\omega].\end{aligned}$$

When the conditions $c_2 = 0$, and $\alpha_0(x) \equiv 0$ are satisfied, then differentiating this expression twice with respect to x yields the given partial differential equation for y. □

Remark 4.1. Note that if $\alpha_0(x)$ is constant (independent of x), then it must be identically zero in order for the boundary conditions to be satisfied. It then follows from (21) that $c_2 = 0$. If on the other hand $\alpha_0(x)$ is not constant, the integral terms in (34) may not be eliminiated by differentiation. Hence, we conclude that the conditions in the theorem are essentially the only ones in which we may expect to find the linearized model of the given rotating planar body beam system given by partial differential equations.

5. Geometrically Exact Linearization

5.1. Geometric effects in rotational equilibrium dynamics

The linearized model (31) includes terms involving the rotation rate ω, reflecting the stiffening effect due to centrifugal forces. If one were to develop a dynamical model of the beam retaining only first-order terms as indicated in the strain expressions (15)–(16) in Section 3 above, the effect of such stiffening would not be captured. This observation motivated the recent work of SIMO & VU-QUOQ [4] who discussed the need to use nonlinear approximations to strain measures. The aim of the present section is to present a more detailed analysis of the consequences of not including higher-order terms in our models. Although, as is suggested in [4], there are implicit constitutive constraints imposed by truncating higher-order terms, it is important to note that the observed effects are of a fundamentally geometric nature and have nothing to do with elasticity theory. Our analysis below provides guidelines for determining how many terms one needs to retain in order to obtain the desired accuracy in the dynamical equations.

Stated in simple terms, the reason that the classical linearization for the beam (i.e., the Euler-Bernoulli theory) does not capture stiffening effects is that this linearization is carried out in terms of displacements from the undeformed position of the elastic structure, and not in terms of its deflection from its non-trivial equilibrium shape. Since it may be difficult to solve a nonlinear differential-integral equation of the form (21) to obtain the equilibrium shape, it is generally not straightforward to write down an exact linearization. An alternative approach, tacitly used in [4], is to linearize about a high-order approximation to the equilibrium. For systems depending on a parameter, where an exact equilibrium is known for a particular value of the parameter, we adopt the premise that an arbitrarily good approximation to the equilibrium solution may be obtained by retaining terms of sufficiently high order in an expansion in terms of the parameter about the given equilibrium. We elucidate this point in a conceptual way first, and then provide an illustrative example.

Consider an elastic structure whose exact equations of motion are written in the form

$$F(U, \Lambda) = 0;$$

for instance, we could have $U = (u, \dot{u}, \ddot{u})$, where $u = u(x, t)$ may denote the displacement vector of the line of centroids of a beam, or the position of the particles of a string, while Λ is a collection of parameters. In our example below it will correspond to the angular velocity of the rotation of the system.

We denote the equilibrium position (i.e., the solution with $\dot{U} = 0$) corresponding to the parameter Λ by U_Λ; without loss of generality we assume that $U_0 = 0$, i.e., that the undeflected position of the structure is in equilibrium for $\Lambda = 0$.

Wishing to obtain the linearized equations, we introduce the deflection V from the equilibrium: $V = U - U_\Lambda$; the equations are now of the form

$$F(U_\Lambda + V, \Lambda) = 0.$$

Expanding in Taylor series in V at $V = 0$, neglecting quadratic and higher order terms, and using the equilibrium relation $F(U_\Lambda, \Lambda) = 0$ we obtain the linearized equation near U_Λ

$$F_U(U_\Lambda, \Lambda)\, V = 0. \tag{35}$$

Expanding near (0, 0) we obtain

$$[F_U(0, 0) + F_{UU}(0, 0)\, U_\Lambda + F_{U\Lambda}(0, 0)\, \Lambda + \tfrac{1}{2} F_{U\Lambda\Lambda}\Lambda^2 + \ldots]\, V = 0,$$

where more (or fewer) terms could be retained. We assume that not only V, but also Λ and thus U_Λ are small, although we do not make any assumptions on the relative size of U_Λ and V. (Typically $\|V\| \ll \|U_\Lambda\| \ll 1$.) This assumption justifies the omission of quadratic and higher order terms in U_Λ. We retain the term involving Λ^2 because in the example below this is the lowest order *nonzero* term in Λ. To find the equilibrium position U_Λ, we differentiate the identity $F(U_\Lambda, \Lambda) = 0$ with respect to Λ to obtain $U_\Lambda = -(F_U)^{-1} F_\Lambda \Lambda + \ldots$, which results in the linearized equation for V:

$$\{F_U(0, 0) + G_1\Lambda + G_2\Lambda^2\}\, V = 0. \tag{36}$$

where $G_1 = F_{U\Lambda}(0, 0) - F_{UU}(0, 0)\, F_U(0, 0)^{-1}\, F_\Lambda(0, 0)$ and G_2 is an expression involving partial derivatives of F of order less than or equal to three evaluated at $(U, V) = (0, 0)$.

It should be observed that (36) differs significantly from the linearized equation $F_U(0, 0)\, W = 0$ through terms of order 1 and 2 in Λ. Depending on the physical problem one may wish to retain more or fewer terms in the above expansions, and by retaining enough terms, any desired degree of accuracy can be achieved. In a rotating structure we would like to keep terms quadratic in ω. It must be emphasized that the orders of magnitude of V and of Λ are unrelated, and in some examples it is reasonable to retain quadratic (and maybe higher-order) terms in Λ while retaining only linear terms in V. Precisely this situation is described below.

5.2. *The example of a rotating string*

The treatment above is completely general; we present an example illustrating these ideas. Consider a string fixed at two points on a platform rotating with constant angular velocity ω. Let one of these points be at the center and another at a distance σl, $\sigma > 1$, where l denotes the natural length of the string. Assuming that the string's tension is proportional to its elongation, we express the potential energy by

$$V = \frac{k}{2} \int_0^l (|u_x| - 1)^2 \, dx \equiv \int_0^l \mathcal{V}(u_x) \, dx, \qquad \mathcal{V} = \frac{k}{2} (|u_x| - 1)^2,$$

where $u = u(x, t) = (u_1(x, t) + x, u_2(x, t))^T$ denotes the position of a point on the string whose undeformed position is along the radial x-axis and is measured by the length x. The equations of motion of this system can be derived directly from the Lagrangian formalism or looked up in [1]; they are of the form

$$\ddot{u} - \omega^2 u - 2i\omega\dot{u} = -\frac{\delta V}{\delta u}.$$

Here $(\dot{\cdot})$ denotes $\frac{\partial}{\partial t}(\cdot)$, $i = \sqrt{-1}$ is used to express the Coriolis force which is perpendicular to the velocity, and the term on the right-hand side is the Fréchet derivative of the potential energy, giving the force (density). These are exact equations of motion *for the model.* Calculating the Fréchet derivative of the potential energy in terms of the integrand $\mathcal{V}$, we write out the dynamical equations in components:

$$\ddot{u}_1 - \omega^2(x + u_1) + 2\omega\dot{u}_2 = \frac{\partial^2 \mathcal{V}}{\partial u_{1x}^2} u_{1xx} + \frac{\partial^2 \mathcal{V}}{\partial u_{1x}\,\partial u_{2x}} u_{2xx}, \tag{37}$$

$$\ddot{u}_2 - \omega^2 u_2 - 2\omega\dot{u}_1 = \frac{\partial^2 \mathcal{V}}{\partial u_{2x}^2} u_{2xx} + \frac{\partial^2 \mathcal{V}}{\partial u_{1x}\,\partial u_{2x}} u_{1xx}, \tag{38}$$

where u_{1xx} denotes the second partial derivative of u with respect to x.

System (37)–(38) is in the form $F(U, \Lambda) = 0$, where F takes values in $\mathbb{R}^2$, $U = (u, \dot{u}, \ddot{u})$ and $\Lambda = \omega$. For any equilibrium solution we have $u_2 \equiv 0$; using this we compute the derivatives of $\mathcal{V}$ at an equilibrium:

$$\frac{\partial^2 \mathcal{V}}{\partial u_{1x}^2} = k, \qquad \frac{\partial^2 \mathcal{V}}{\partial u_{2x}^2} = k\left(\frac{u_{1x}}{1 + u_{1x}}\right), \qquad \frac{\partial^2 \mathcal{V}}{\partial u_{1x}\,\partial u_{2x}} = 0.$$

We consider first the case of $\omega = 0$, and linearize the equations around the obvious equilibrium solution $u_1 = (\sigma - 1)\,x$, $u_2 \equiv 0$. The linearized equations are of the form

$$\ddot{v}_1 - kv_1'' = 0, \tag{39}$$

$$\ddot{v}_2 - cv_2'' = 0, \tag{40}$$

with $c = \dfrac{\partial^2 \mathscr{V}}{\partial u_{2x}^2} = k\left(1 - \dfrac{1}{\sigma}\right) > 0$ and with $'$ denoting the x-derivative. These equations describe the longitudinal and transversal oscillations of the string and agree with the standard equations for the vibrating string. Here we have to point out that c is an increasing function of the parameter σ, which controls the stretching of the string. In particular, the frequencies of the transversal oscillations will grow with σ, a familiar effect in string instruments. This is precisely the static stiffening effect, caused in this case by the boundary conditions.

For nonzero angular velocities, the stationary solution still has zero transversal displacement $u_2 \equiv 0$, and the expressions given in (37)–(38) for the inertial and potential forces still hold. In order to linearize the equations we have to compute the equilibrium solution, which in the previous case was trivial. Instead of using the abstract formula (36), we write down an exact expression and then examine the approximation. This approach is not always feasible, but since our example allows an exact solution, we give it.

It is not difficult to see that (37), (38) admit equilibrium solutions in which the transversal displacement vanishes ($u_2 \equiv 0$), while the longitudinal displacement satisfies the equation

$$k u_1'' + \omega^2(x + u_1) = 0,$$

which we solve using the boundary conditions to obtain

$$u_1^\omega = \sigma l \frac{\sin \dfrac{\omega}{\sqrt{k}} x}{\sin \dfrac{\omega}{\sqrt{k}} l} - x.$$

Linearizing the original system around this solution we obtain

$$\ddot{v}_1 - \omega^2 v_1 + 2\omega \dot{v}_2 - k v_1'' = 0, \tag{41}$$

$$\ddot{v}_2 - \omega^2 v_2 - 2\omega \dot{v}_1 - (c(x)\, v_2')' = 0, \tag{42}$$

where $c(x) = k\left(1 - \dfrac{\sqrt{k}}{\sigma l \omega} \dfrac{\sin \dfrac{\omega}{\sqrt{k}} l}{\cos \dfrac{\omega}{\sqrt{k}} x}\right)$. As one would expect, there is a critical velocity $\omega_c = \pi \sqrt{k}/2l$ such that for $\omega \geqq \omega_c$, the function $c(\cdot)$ has singularities.

In writing down (41) and (42), we have found the exact equilibrium solution corresponding to U_Λ in the abstract formulation. We can retain any desired accuracy in Λ (which specializes to ω in this case) by keeping an appropriate number of terms in the Taylor expansion of $c(x)$ in terms of ω. For instance,

$$c(x) = k\left(1 - \frac{1}{\sigma}\right) + \frac{1}{\sigma}\left(\frac{x^2}{2} - \frac{l^2}{6}\right)\omega^2 + \dots .$$

Note that there are no first-order terms in ω. The quadratic and higher-order terms in this expression reflect the way centrifugal stiffening affects the string dynamics. For the specific value $\omega = \omega_c$, Figure 6 compares an approximation for $c(\cdot)$ second-order (in ω) with the corresponding exact function. As ω increases and approaches ω_c, the quadratic approximation degrades, especially for $x \approx l$, reflecting a need to retain terms of even higher order in ω. An indication of the quality of the quadratic approximation is provided in Figure 7 for several values of x between 0 and l.

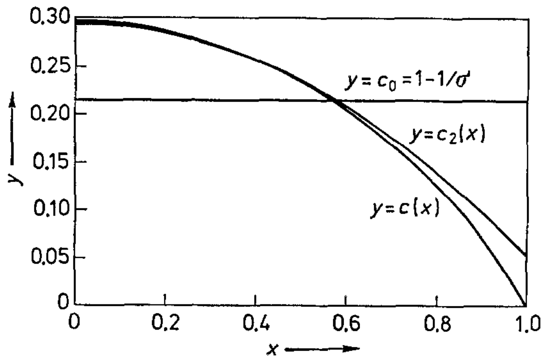

Fig. 6. $c(x)$ and its approximations of order 0 and 2 (in ω) evaluated at $\omega = \omega_c/2$.

The point we wish to emphasize is that we have linearized about an *equilibrium solution* in this example, not about the origin. As ω takes on larger values, the quadratic and higher-order terms (in ω) which appear in the stiffening coefficient $c(x)$ become increasingly significant in the model.

6. The Dynamics of the Rotating Pendulum and Constrained Relative Motions in Multibody Systems

Having described various models of rotating mechanical systems in Sections 2 through 5, and having discussed a variety of ways in which local constraints on relative motions affect the global behavior of rotating mechanical systems, we shall now examine this phenomenon in the simple context of a rotating pendulum with a single link. This example is motivated in part by our earlier work on rotating (planar) kinematic chain models [24]. In these models, it is assumed that the motion of the kinematic chains are confined to a plane which is rotated about a vertical axis (the neutral axis of the chain) in a gravitational field. In light of the discussion above, it is of interest to ask how the qualitative features of these models would change if motions were not constrained rigidly to the rotating plane.

Consider a mechanism consisting of a solid uniform rectangular parallelepiped fixed at one end to a universal joint as depicted in Figure 8. The universal joint is composed of two single-degree-of-freedom revolute joints with mutually orthogonal intersecting axes. We may choose a coordinate frame of principal axes (with respect to the rectangular mass) centered at the intersection of the two joint axes; we further assume that each of the two joint axes is aligned parallel to one of the

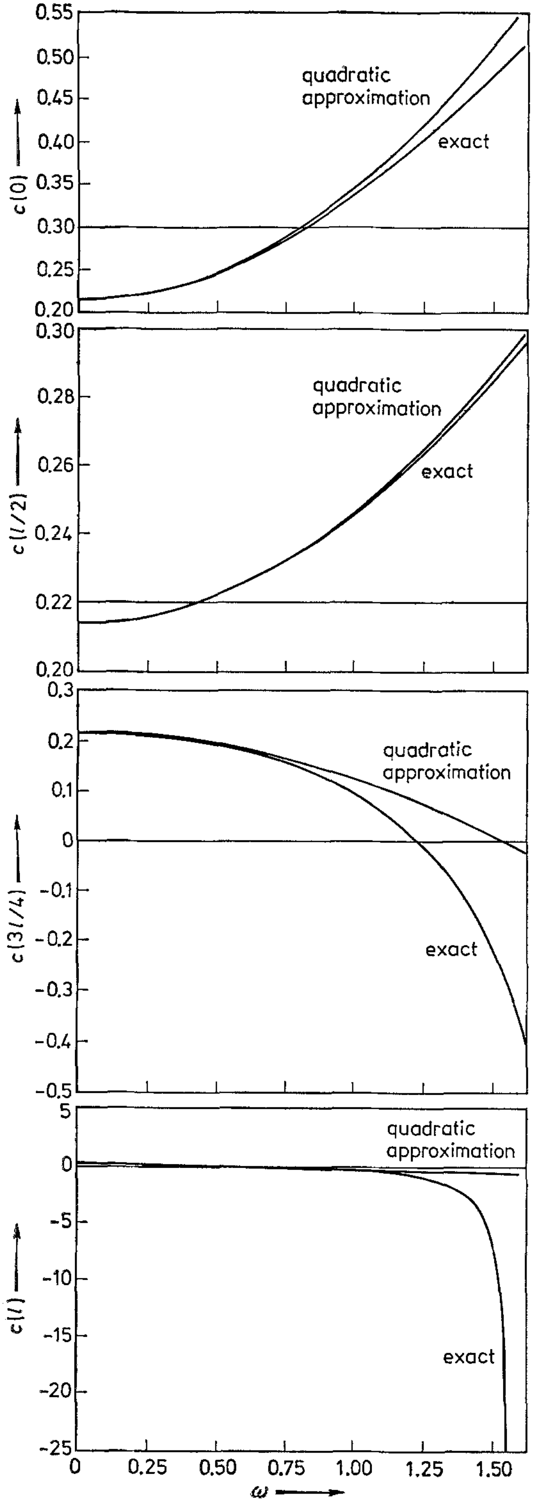

Fig. 7. Plots of the dependence of $c(x)$ on ω at $x = 0$, $l/2$, $3l/4$, and l.

(principal) coordinate axes, and the third coordinate axis is perpendicular to the joint axes. We designate the two joint axes x and y with the remaining axis labeled z. These axes define a 'body frame' and are depicted in Figure 8. Also depicted in Figure 8 is a third joint whose axis of rotation is fixed with respect to a fixed (inertial) frame of reference. Let θ measure the angle of rotation about this axis with $\theta = 0$ defining an arbitrarily chosen reference. In the absence of external forcing, the pendulum is aligned with gravity, having its jointed end 'up.' In such a configuration with $\theta = 0$, the gravitational potential is minimized, and the 'body frame' coincides with an inertially fixed cordinate frame — with axes denoted by x', y', and z' in Figure 8 — with respect to which we shall measure all motion. We assume there is an actuator which is capable of forcing the mechanism to rotate about the inertial z-axis with any prescribed angular velocity.

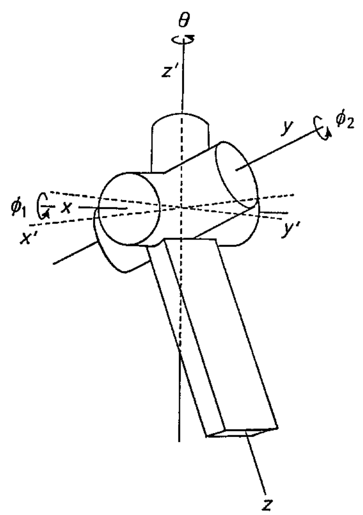

Fig. 8. A rotating pendulum suspended from a universal joint.

With θ denoting the angle of rotation about the (inertial) z-axis, let ϕ_1 and ϕ_2 denote the angles of rotation about each of the (principal body) axes x and y respectively. Let I_x, I_y, and I_z denote the principal moments of inertia with respect to the body coordinate system. Then the system has the Lagrangian

$$L(\dot{\theta}; \phi_1, \dot{\phi}_1; \phi_2, \dot{\phi}_2) = \tfrac{1}{2}\,(I_x(\dot{\theta}s_2 + \dot{\phi}_1)^2 + I_y(\dot{\theta}c_2 s_1 - \dot{\phi}_2 c_1)^2$$

$$+ I_z(\dot{\theta}c_1 c_2 + \dot{\phi}_2 s_1)^2) + c_1 c_2, \tag{43}$$

where the last term is a normalized gravitational potential and where $s_i = \sin\phi_i$, $c_i = \cos\phi_i$. We shall study the dynamics of this system in the case that $I_x \geqq I_y \gg I_z$, and as in [24], we shall view $\dot{\theta}(\cdot)$ as a control input. Of particular interest will be relative equilibrium solutions satisfying $\dot{\phi}_i \equiv 0$ and the constraint that $\dot{\theta} = \omega \equiv$ const. Viewing ω as a parameter, we define

$$\tilde{L}(\omega; \phi, \dot{\phi}) = \tfrac{1}{2} (I_x\dot{\phi}_1^2 + (I_y c_1^2 + I_z s_1^2)\, \dot{\phi}_2^2 + ((I_z c_1^2 + I_y s_1^2)\, c_2^2 + I_x s_2^2)\, \omega^2) + I_x s_1 \omega \dot{\phi}_1 + (I_z - I_y)\, s_1 c_1 c_2 \omega \dot{\phi}_2 + c_1 c_2, \tag{44}$$

which is in effect the *reduced Lagrangian* obtained by reducing the original Lagrangian (43) with respect to the $SO(2)$ symmetry defined by rotation about the inertial z-axis. (Cf. [25], Ch. 3.) Note that the original Lagrangian L is composed only of terms of order 2 and order 0 in the velocities, whereas the reduced Lagrangian $\tilde{L}$ also includes terms which are linear in the velocities $\dot{\phi}_1$ and $\dot{\phi}_2$ (with ω viewed as a parameter). The relative equilibria corresponding to $\dot{\theta} \equiv \omega$ are then found as the standard equilibria, $\dot{\phi}_i \equiv 0$, of the *reduced dynamics*

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\phi}_i} - \frac{\partial \tilde{L}}{\partial \phi_i} = 0 \tag{45}$$

or, more generally, if dissipation is included, of the reduced dynamics

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\phi}_i} - \frac{\partial \tilde{L}}{\partial \phi_i} = -d_i \dot{\phi}_i \tag{46}$$

where $d_i \geqq 0$.

Setting $\dot{\phi}_1 = \dot{\phi}_2 = 0$ in (45) or (46) for $i = 1, 2$ yields

$$((I_z - I_y)\, \omega^2 c_1 c_2 + 1)\, s_1 c_2 = 0, \tag{47}$$

$$((I_z c_1^2 + I_y s_1^2 - I_x)\, \omega^2 c_2 + c_1)\, s_2 = 0. \tag{48}$$

While the recent literature (e.g., [24], [16], [26], [18]) on the equilibrium dynamics of rotating systems suggests the existence of a rich parameter-dependent set of solutions to (47) and (48), we shall at present avoid the full complexity of this problem by restricting our attention to solutions within the range $|\phi_i| < \frac{\pi}{2}$. Away from the boundary of this region, our mechanism provides a kinematic approximation of a rotating spherical pendulum. Within the prescribed region, the next proposition describes the stability and bifurcations of the relative equilibria.

Proposition 6.1. *For* $I_x = I_y \gg I_z$, *the solutions to* (47), (48) *within the range* $|\phi_i| < \frac{\pi}{2}$ $(i = 1, 2)$ *have the following bifurcation and stability properties.*
(i) *If* $(I_x - I_z)\, \omega^2 < 1$, *the only solution to* (47)–(48) *is* $(\phi_1, \phi_2) = (0, 0)$. *With respect to the reduced dynamics* (46), *this equilibrium is stable.*
(ii) *If* $(I_x - I_z)\, \omega^2 > 1$, *then* $(\phi_1, \phi_2) = (0, 0)$ *is an unstable equilibrium. In addition, there is a one-dimensional set of* (*linearly*) *stable relative equilibria which satisfy* $(I_z - I_x)\, \omega^2 c_1 c_2 + 1 = 0$.

For $I_x > I_y \gg I_z$, *there are* 1, 3, *or* 5 *solutions to* (47), (48) (*counted according to multiplicity*) *with* $|\phi_i| < \frac{\pi}{2}$. *They have the following properties.*

(iii) *If* $\omega^2 < (I_x - I_z)^{-1}$, *the only solution to* (47)–(48) *is* $(\phi_1, \phi_2) = (0, 0)$. *With respect to the reduced dynamics* (46) *this solution is stable.*
(iv) *If* $(I_x - I_z)^{-1} < \omega^2 < (I_y - I_z)^{-1}$, *then* $(\phi_1, \phi_2) = (0, 0)$ *remains a solution to* (47)–(48) *which is an unstable equilibrium for the reduced dynamics* (46). *In addition, there are two solutions to* (47), (48) *which also satisfy* $\phi_1 = 0$ *and* $(I_z - I_x)\,\omega^2 c_2 + 1 = 0$. *These two solutions are stable equilibria of* (46) $(i = 1, 2)$.
(v) *If* $(I_y - I_z)^{-1} < \omega^2$, *there are in addition to the equilibria of the preceding two cases two solutions to* (47)–(48) *defined by* $\phi_2 = 0$ *and* $(I_z - I_y)\,\omega^2 c_1 + 1 = 0$. *The stability properties of the equilibria for this range of parameters are*: $(\phi_1, \phi_2) = (0, 0)$ *is locally unstable*; *the two equilibria defined by* $\phi_1 = 0$ *and* $(I_z - I_x)\,\omega^2 c_2 + 1 = 0$ *are stable*; *the two equilibria defined by* $\phi_2 = 0$ *and* $(I_z - I_y)\,\omega^2 c_1 + 1 = 0$ *are unstable.*

Proof. Assertions regarding the stability of the equilibria $(\phi_1, \phi_2) = (0, 0)$ (when $\omega^2 < (I_x - I_z)^{-1}$) and $(\phi, \phi_2) = (0, \arccos(1/\omega^2(I_x - I_z))$ (when $I_x > I_y$ and $\omega^2 > (I_x - I_z)^{-1}$) may be established by an energy argument as follows. Even though the equations (45) represent the dynamics of a forced or driven system, a straightforward computation shows that the *reduced* energy function

$$\tilde{E} = \tfrac{1}{2}\,(I_x\dot{\phi}_1^2 + (I_y c_1^2 + I_z s_1^2)\,\dot{\phi}_2^2) - \tfrac{1}{2}\,((I_z c_1^2 + I_y s_1^2)\,c_2^2 + I_x s_2^2)\,\omega^2 - c_1 c_2$$

is a conserved quantity. In fact, this is not surprising, since $\tilde{E}$ is related to the reduced Lagrangian by means of a Legendre transformation: $\tilde{E} = \dfrac{\partial \tilde{L}}{\partial \dot{\phi}} \cdot \dot{\phi} - \tilde{L}$.

If $\omega^2 < (I_x - I_z)^{-1}$, then $(\phi, \dot{\phi}) = (0, 0)$ is a local minimum of $\tilde{E}$, and it follows from standard results (e.g., [27], pp. 172–173) that $(\phi, \dot{\phi}) = (0, 0)$ is a stable equilibrium of (45). If $I_x > I_y$, then as ω^2 increases through the value $(I_x - I_z)^{-1}$, this equilibrium undergoes a pitchfork bifurcation, and for $\omega^2 > (I_x - I_z)^{-1}$, $(\phi, \dot{\phi}) = (0, 0)$ is no longer a local minimum of $\tilde{E}$. The symmetric pair of equilibria defined by $\phi_1 = 0$ and $(I_z - I_x)\,\omega^2 c_2 + 1 = 0$ are local minima for all $\omega^2 > (I_x - I_z)^{-1}$, and are thus stable.

All assertions regarding the instability of various equilibria may be established by carefully analyzing the linearized dynamics. When $I_x > I_y$ and $(I_x - I_z)^{-1} < \omega^2 < (I_y - I_z)^{-1}$, the linearization of (45) about the equilibrium $(\phi, \dot{\phi}) = (0, 0)$ has an eigenvalue in the right half-plane, and thus it is unstable. When $I_x = I_y$ or $\omega^2 > (I_y - I_z)^{-1}$, the linearization of (45) about $(0, 0)$ has only imaginary eigenvalues, and a linear stability analysis is conclusive only with dissipation present (as in (46)). We omit the details of this analysis, although it is pursued to some extent below to reveal other qualitative features. When $I_x > I_y$ and $\omega^2 > (I_y - I_z)^{-1}$, the linearization of (45) about the equilibrium $\phi_2 = 0$, $(I_z - I_y)\,\omega^2 c_1 + 1 = 0$ also has an eigenvalue in the right half-plane.

Finally, we note that in the special case that $I_x = I_y$ and $\omega^2 > (I_x - I_z)^{-1}$, the matrix of second partial derivatives of $\tilde{E}$ is singular at each member of the family of equilibria determined by the equation $(I_z - I_x)\,\omega^2 c_1 c_2 + 1 = 0$. For these equilibria, we can expect only linear stability. Not surprisingly, there is always a zero eigenvalue in the linearization of (45) or (46). □

Remark 6.1. A graphical interpretation of Proposition 6.1 is given in Figure 9. For $I_x > I_y$, there is a sequence of pitchfork bifurcations of equilibria of (45) as the parameter ω^2 increases through the values $(I_x - I_z)^{-1}$ and $(I_y - I_z)^{-1}$. At the first bifurcation, $\omega^2 = (I_x - I_z)^{-1}$, a symmetric pair of stable equilibria bifurcate from $(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = (0, 0, 0, 0)$, which in turn becomes an unstable equilibrium with a one-dimensional unstable manifold. At the second bifurcation, $\omega^2 = (I_y - I_z)^{-1}$, a symmetric pair of equilibria with one-dimensional unstable manifolds bifurcates from the same $(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = (0, 0, 0, 0)$, which then becomes unstable with a two-dimensional unstable manifold.

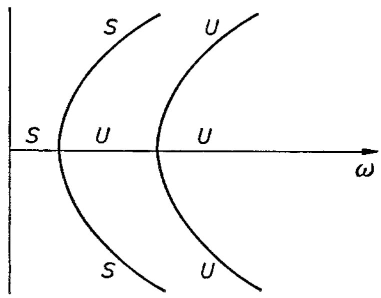

Fig. 9. Bifurcation diagram of Proposition 6.1.

Remark 6.2. While the above energy arguments using $\tilde{E}$ prove nonlinear stability and are appealingly straightforward and consistent with a general approach used by other authors (cf. [3], [7], and [19]), a more refined understanding of asymptotic stability is obtained by studying the linearization of (46) about each equilibrium. In the case $(\phi_1, \phi_2) = (0, 0)$, the first-order variational equations are

$$\ddot{x} + \beta_1 \dot{x} + (1 + \alpha_1)\,\omega \dot{y} + (\gamma_1 - \alpha_1 \omega^2)\,x = 0, \tag{49}$$

$$\ddot{y} + \beta_2 \dot{y} - (1 + \alpha_2)\,\omega \dot{x} + (\gamma_2 - \alpha_2 \omega^2)\,y = 0 \tag{50}$$

where $x = \delta\phi_1, y = \delta\phi_2$, $\alpha_1 = (I_y - I_z)/I_x$, $\alpha_2 = (I_x - I_z)/I_y$, $\beta_1 = d_1/I_x$, $\beta_2 = d_2/I_y$, $\gamma_1 = 1/I_x$ and $\gamma_2 = 1/I_y$. These equations together define a fourth-order system with characteristic equation

$$\lambda^4 + (\beta_1 + \beta_2)\,\lambda^3 + (\beta_1\beta_2 + \gamma_1 + \gamma_2 + (1 + \alpha_1\alpha_2)\,\omega^2)\,\lambda^2 + (\beta_1(\gamma_2 - \alpha_2\omega^2) + \beta_2(\gamma_1 - \alpha_1\omega^2))\,\lambda + (\gamma_1 - \alpha_1\omega^2)\,(\gamma_2 - \alpha_2\omega^2) = 0. \tag{51}$$

If we set the damping $\beta_i = 0$, then under our assumption that $I_x \geqq I_y \gg I_z$, it follows that the characteristic equation may always be factored into

$$(\lambda^2 + r_1)\,(\lambda^2 + r_2) = 0, \tag{52}$$

where r_1 and r_2 are real numbers (depending on the parameters α_j, β_j, γ_j). Note that $\gamma_2/\alpha_2 \leqq \gamma_1/\alpha_1$, with equality holding if and only if $I_x = I_y$. When $\omega^2 < \gamma_2/\alpha_2$, both r_1 and r_2 are positive, and all roots of (51) are purely imaginary. Thus the linear dynamics (49), (50) do not determine the stability of the equilibrium, but as indicated in the proof of Proposition 6.1, an energy argument shows that we do in fact have (nonlinear) stability in this case. When $\gamma_2/\alpha_2 < \omega^2 < \gamma_1/\alpha_1$, r_1 and r_2 have opposite signs. There is one positive root of (51), and there is thus a range of velocities ω which make the linearized dynamics unstable. When $\omega^2 > \gamma_1/\alpha_1$, we see that once again all roots of the characteristic equation are imaginary, and again stability of the equilibrium may not be determined from the linearized dynamics. On the other hand, if there is damping $\beta_i > 0$ for $i = 1, 2$, it is easy to construct the Routh table showing the existence of an eigenvalue in the right half-plane for the linearized dynamics. The point we wish to emphasize is that even though $(\phi, \dot{\phi}) = (0, 0)$ is an unstable equilibrium when $\omega^2 > \gamma_1/\alpha_1$, the eigenvalues of the linearization will nevertheless be close to the imaginary axis for sufficiently small values of the damping parameters.

We conclude by contrasting the stability analysis for this rotating pendulum suspended from a two-degree-of-freedom universal joint with corresponding results for a one-degree-of-freedom rotating simple pendulum. To simplify these remarks it will be useful to assume $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$. If in equation (45) for $i = 1$, we set $\phi_2 = \dot{\phi}_2 = 0$, we obtain an equation describing the dynamics of a simple pendulum undergoing forced rotation about an axis passing through its joint and lying in the plane of its motion. The dynamics of multilink kinematic chains of this form were studied in [24]. A general feature of such planar kinematic chains undergoing forced rotation at a constant angular velocity ω about the vertical axis is that the dynamics may be derived from a reduced Lagrangian of the form

$$\tilde{L}_1(\omega; \phi, \dot{\phi}) = \tfrac{1}{2}\, \dot{\phi}^T M_1(\phi)\, \dot{\phi} - V_1(\omega; \phi). \tag{53}$$

We note that this stands in contrast to the case of chains in which there is *more* than a single degree of freedom in the relative motions between links. If the successive links are coupled, say, by universal joints as in the single link mechanism described above, the dynamics of the chain, forced at constant angular velocity ω about the vertical axis, is derived from the reduced Lagrangian

$$\tilde{L}_2(\omega; \phi, \dot{\phi}) = \tfrac{1}{2}\, \dot{\phi}^T M_2(\phi)\, \dot{\phi} + \omega A(\phi)\, \dot{\phi} - V_2(\omega; \phi). \tag{54}$$

(Cf. equation (44).) The key difference between these two situations is that when the chain links are not confined to move in a (rotating) plane, the Lagrangian includes Coriolis or *gyroscopic* terms, $\omega A(\phi)\, \dot{\phi}$, which are linear in the velocities $\dot{\phi}$. Some of the differences between the qualitative dynamics associated with (53) and (54) may be gleaned by comparing the linearization (49), (50) with the corresponding linearization about $(\phi, \dot{\phi}) = (0, 0)$ of the single-degree-of-freedom rotating pendulum:

$$\ddot{x} + \beta\dot{x} + (\gamma - \alpha\omega^2)\, x = 0. \tag{55}$$

In (55) there is an eigenvalue in the right half-plane for all values of the damping parameter $\beta \geqq 0$ precisely when $\alpha\omega^2 > \gamma$. By choosing $\omega^2 > \gamma/\alpha$ large enough, this positive real eigenvalue may be made as large as we like, whereas for the same ω and sufficiently small values of the damping parameter β, the eigenvalues of (51) in the right half-plane may be made to lie arbitrarily close to the imaginary axis. We have thus obtained the somewhat paradoxical result that, at least in the lightly damped case, the more highly constrained system is less stable than the one with greater freedom.

7. Concluding Remarks

While a complete theory of the way in which the global dynamics of a complex mechanical system are affected by constraints on the relative motions of component parts remains to be fully articulated, the outline of such a theory can be glimpsed in the preceding discussion. The examples of Sections 2 and 6 illustrate how both steady-state and transient dynamics of rotating systems may be dramatically changed by constraining relative motions. In exploring this circle of ideas in the context of rod mechanics in Section 3, it was shown how constitutive restrictions affect the form of the models and give rise to qualitative differences in dynamic equilibria of planar rotating body-beam systems. In this setting, it is interesting to note that certain modeling assumptions wherein small terms of high order are ignored (as in the derivation of Euler-Bernoulli beam models) are equivalent to imposing constitutive restrictions of the type we have considered explicitly. While it is not generally straightforward to characterize the geometric or kinematic consequences of ignoring higher-order terms, the rotating string example analyzed in Section 5 does give a rather clear indication of the role such terms play in capturing the dynamical effect of centrifugal stiffening.

We conclude by noting that the comparison of the one- and two-degree-of-freedom dynamics in Section 6 indicates, as one would expect, that gyroscopic forces (which are present in the two-degree-of-freedom case) have a significant stabilizing effect. For kinematic chains with more degrees of freedom, recent laboratory experiments (to be reported elsewhere) also indicate that instabilities are less severe when the relative motions between the links have more than a single degree of freedom. Various applications have stimulated a recent renewal of interest in the dynamics of systems which are subject to gyroscopic forces (see, e.g., [28]). The role of such forces in the dynamics and control of rotating kinematic chains will be treated in a future paper.

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