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Vol. 50, No. 3, December 1983
Printed in Belgium

On Stability of Symplectic Maps

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Received April 12, 1982

The aim of this short note is to give a simple geometrical proof of a result due to Cushman and Kelley [4] giving a characterization of strongly stable symplectic maps. The original proof relied on the use of normal forms.

The following is a slight reformulation of the main result of [4].

THEOREM. *An infinitesimally stable*[1] *symplectic matrix A is strongly stable iff its centralizer $C(A)$ (in the set* $\mathrm{sp}(n)$ *defined below) consists of stable matrices.*

We recall first some definitions (see [3–7, 9, 10, 12]). Any $2n \times 2n$ matrix of the form

$$A = JH, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad H^T = H$$

is called infinitesimally symplectic; let $\mathrm{sp}(n)$ be the set of such matrices. Any matrix A is called *stable* iff $\|e^{At}\|$ is bounded for all (positive and negative) t. Matrix $A = JH \in \mathrm{sp}(n)$ is called *strongly stable* if any matrix $B = JK$ with $K = K^T$ sufficiently close to H is stable.

Centralizer $C(A)$ of a matrix A is, by definition, the set of all matrices in $\mathrm{sp}(n)$ commuting with A.

Before proceeding with the proof, we will need one perturbation result [4, 1].

LEMMA. *Any matrix $B \in \mathrm{sp}(n)$ sufficiently close to a stable matrix $A \in \mathrm{sp}(n)$ can be expressed as*

$$B = S^{-1}(A + C)\,S \qquad \text{with} \quad C \in C(A),\ S = e^T,\ T \in \mathrm{sp}(n).$$

* Supported in part by the NSF Grant MCS-8212681. Present address: Department of Mathematics, Boston University, Boston, Massachusetts 02215.

[1] Definitions follow the statement of this theorem.

0022-0396/83 $3.00

Proof of the Theorem (followed by the proof of the lemma). 1. Assume that $C(A)$ consists of stable matrices. Any $B \in \mathrm{sp}(n)$ close to A can be written as

$$B = S^{-1}(A + C)\,S, \qquad C \in C(A),$$

according to the above lemma. Therefore, B is stable, being similar to stable matrix $A + C \in C(A)$; strong stability of A is proven.

2. Conversely, assume that A is strongly stable; choose any $B \in C(A)$ and show its stability. For ε small enough we have, for some $c > 0$,

$$\|e^{-At}\|, \qquad \|e^{(A+\varepsilon B)t}\| < c \text{ for all } t,$$

since A is strongly stable. Using the fact that A and B commute ($B \in C(A)$), we have

$$\|e^{\varepsilon Bt}\| = \|e^{-At}e^{(A+\varepsilon B)t}\| < c^2,$$

which proves stability of B. Q.E.D.

Proof of the Lemma. Introduce a map

$$M : \mathrm{ran\ ad}_A \oplus \ker \mathrm{ad}_A \to \mathrm{sp}(n),$$

given by

$$M(T, C) = e^{-T}(A + C)\,e^{T};$$

here $\mathrm{ad}_A X = [A, X]$. Wishing to apply the implicit function theorem to M near $T = C = 0$, we calculate its derivative:

$$DM(0,0)(T, C) = [A, T] + C \in \mathrm{ran\ ad}_A \oplus \ker \mathrm{ad}_A = \mathrm{sp}(n).$$

The last equality follows from the fact that ad_A is semisimple (i.e., diagonalizable), which in turn is the consequence of stability (and thus diagonalizability) of A.

This shows that $DM(0,0)$ maps $\mathrm{ran\ ad}_A \oplus \ker \mathrm{ad}_A$ onto itself; by the implicit · function theorem for any $B \in \mathrm{sp}(n)$ there exists $T \in \mathrm{ran\ ad}_A$, $C \in \ker \mathrm{ad}_A = C(A)$ with

$$B = M(C, T) = e^{-T}(A + C)\,e^{T} = S^{-1}(A + C)\,S. \qquad \text{Q.E.D.}$$

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium