Particles with Negative Mass and the Krein–Moser Theory

MATHEMATICAL CURIOSITIES

By Mark Levi

Imagining objects with negative mass may seem like a scholastic exercise. But as it turns out, we can interpret some real physical phenomena as the occurrence of particles with negative mass.

As a glimpse of the strange world with negative masses, Figure 1 shows two masses of equal magnitude but opposite sign connected by a Hookean spring.[1] Initially all is at rest and the spring is stretched, pulling the masses towards each other. In response, the positive mass will accelerate to the right as expected; the negative mass,

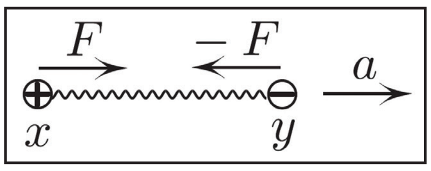

Figure 1. *Two masses of opposite sign connected by a spring. Acceleration happens with no external force applied.*

being pulled to the left, will accelerate against the pull, i.e. to the right as well. So the whole system will accelerate to the right, with the distance between the masses remaining constant. Formally, the positions x and y of our particles satisfy

$$m\ddot{x} = F, \quad -m\ddot{y} = -F,$$

where F is the force of the spring. Addition gives $\frac{d^2}{dt^2}(x-y)=0$, so that the distance $x-y=$constant (since $\dot{x}=\dot{y}$ at $t=0$) as claimed, and subtraction results in

$$\frac{d^2}{dt^2}(x+y)/2 = F/m,$$

showing that indeed the midpoint accelerates at a constant rate F/m.

This acceleration occurs with no external forces applied and does not contradict Newton's second law, since the total mass of the system is zero. The fantastic world of negative masses can have spaceships which require no fuel and no external sources of energy to accelerate in any desired direction (here the discussion is limited to the one-dimensional world, but one can imagine a 3D construction along the lines of Figure 1, with the astronaut pushing or pulling on the right mass and controlling the direction of acceleration.) There is no contradiction with the conservation of energy since the kinetic energy of our system remains zero and the potential energy of the spring remains constant at all times (with the initial conditions as specified).

Consider now the regular harmonic oscillator $\ddot{x}+x=0$, with mass $m=1$ and Hooke's constant $k=1$. Multiplying both sides by -1 yields a mathematically equivalent system $-\ddot{x}-x=0$. Physically, we can interpret this as describing the motion of a particle of negative mass attached to a spring with the negative Hooke's constant. Until such a system is touched, it will behave as a normal mass-spring one. But when connected to a "normal" system—say, by a weak Hookean spring—it may result in an instability. Informally, we can trace the mechanism of this instability to Figure 1; if the positive mass "trails" the negative one, then the interaction will cause *both* masses to accelerate. This is not difficult to see formally on a simple model of two harmonic oscillators connected by a spring with a small Hooke's constant ε:

$$\begin{cases} \ddot{x}+x = \varepsilon(y-x), \\ -\ddot{y}-y = \varepsilon(x-y). \end{cases}$$

We have $\frac{d^2}{dt^2}(x+y)+(x+y)=2(y-x)$, so that $x+y$ behaves as a forced harmonic oscillator. And the forcing $2(y-x)$ satisfies $\frac{d^2}{dt^2}(y-x)+(y-x)=0$, the equation of the harmonic oscillator with the *same frequency* and therefore in resonance with $x+y$, causing the amplitude of oscillations of $x+y$ to grow linearly.

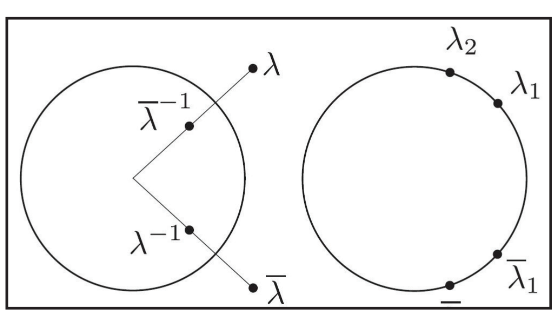

Figure 2. *The Lyapunov-Poincaré theorem: the symmetric spectrum of a symplectic matrix.*

All this is the tip of a very nice theory developed by Mark Krein and Jürgen Moser. In the 1950s, physicists working at Brookhaven National Labs made a puzzling experimental observation; a simple resonance consisting of two frequencies becoming equal led to an instability in some settings but not in others. Moser provided a beautiful explanation of this phenomenon [3], and it turned out that Krein had explained the same phenomenon a few years earlier [1]. The explanation boils down to a beautiful analysis of symplectic matrices; the details can be found in the cited papers of Krein and Moser or in [2].

To give the flavor of the Krein-Moser result, we recall that the spectrum of such a symplectic matrix is symmetric with respect to the unit circle, as illustrated in Figure 2 (this fact is known as the Poincaré-Lyapunov theorem). And thus a simple eigenvalue cannot leave the unit circle under small perturbation of a matrix; otherwise, an extra mirror image eigenvalue would appear. A simple eigenvalue can only leave a circle if it meets another eigenvalue. But not every meeting of eigenvalues causes them to leave the unit circle.

In the Krein-Moser theory, every eigenvalue is assigned a symbol + or – .[2] The "direction of rotation," as measured by the symplectic 2-form, gives the sign. And the beautiful result is that if two same-sign eigenvalues on the unit circle meet (under the deformation of a symplectic matrix), they harmlessly "pass through each other," staying on the unit circle, and thus not causing an instability. But the colliding eigenvalues of opposite sign "bump" each other off of the unit circle, as in Figure 3. And we can interpret these collisions in terms of particles of negative masses connected by springs with negative or positive Hooke's constants. To summarize, the Krein-Moser sign of an eigenvalue can be interpreted as the sign of an imaginary mass.

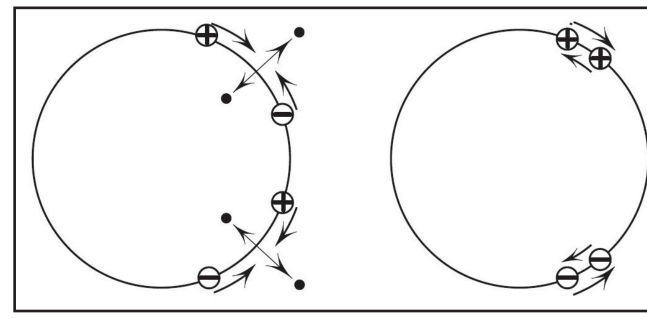

Figure 3. *Collision of the same-sign eigenvalues does not lead to instability; collision of the opposite-sign eigenvalues does.*

[1] In defense of my sanity, I am not suggesting that such masses are real; however, such imaginary masses arise in interpreting real physical systems.

[2] One need not limit the attention to simple eigenvalues, but I want to skip such details.

References

[1] Krein, M.G. (1950) A generalization of some investigations of linear differential equations with periodic coefficients, *Dokl. Akad. Nauk. SSSR N.S., 73*, 445-448.

[2] MacKay, R.S. (1986). Stability of equilibria of Hamiltonian systems. *Nonlinear phenomena and chaos*, 254-270.

[3] Moser, J. (1958). New aspects in the theory of stability of Hamiltonian systems. *Comm. Pure Appl. Math., 11*, 81-114.

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