DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **10**, Number **2&3**, September & October **2008**

Website: http://aimSciences.org
pp. **569–595**

MAKING THE MOON REVERSE ITS ORBIT, OR, STUTTERING IN THE PLANAR THREE-BODY PROBLEM

Samuel R. Kaplan

Department of Mathematics
University of North Carolina at Asheville
Asheville, NC 28804, USA

Mark Levi

Department of Mathematics
Penn State
State College, PA, 16801, USA

Richard Montgomery

Mathematics Department
University of California, Santa Cruz
Santa Cruz, CA, 95064, USA

Abstract. We investigate the planar three-body problem in the range where one mass, say the 'sun' is very far from the other two, call them 'earth' and 'moon'. We show that "stutters" : two consecutive eclipses in which the moon lies on the line between the earth and sun, occur for an open set of initial conditions. In these motions the moon reverses its sense of rotation about the earth. The mechanism is a kind of tidal torque (see the 'key equation'). The motivation is to better understand the limits of variational methods. The methods of proof are classical estimates and bounds in this asymptotic regime.

1. **Introduction.** Consider the planar three-body problem with Newtonian force law. A *syzygy* is a collinear configuration of the three masses. Double collisions count as syzygies, but triple collisions do not count. The rationale behind this counting is that we can analytically families of solutions through binary collisions [8] but not through triple collision. Non-collision syzygies come in three flavors, 1, 2, and 3 , depending on which of the three masses, 1, 2, or 3, lies between the other two at the moment of syzygy. We associate to any non-collinear, collision-free solution its sequences of syzygies, listed in the time order of appearance. For example, a "tight binary" in which 1 and 2 move in a bound, nearly Keplerian orbit while 3 moves far away will execute the syzygy sequence $\ldots 121212\ldots$. A recent theorem of one of us [14] asserts that when the energy is negative and the angular momentum is zero then every solution has syzygies, with the single exception of the Lagrange homothety solution. So syzygies are ubiquitous in this case.

A *stutter* is a syzygy sequence in which the same letter repeats itself, one or more times in a row: for example: 112. From a variational perspective, stutters are

2000 *Mathematics Subject Classification.* Primary: 70F07, 37N05; Secondary: 70H33.
Key words and phrases. Three-body problem, syzygy, orbit reversals.
This work supported in part by NSF grant DMS-20030177 and DMS-0605878 (ML).

perverse within the zero angular momentum problem. (See the next section.) We would like to know how prevalent stutters are. Certainly there are some.

Example 1.1. A "brake orbit" is a solution such that at some instant t_0 all three velocities are zero. Brake orbits have angular momentum zero and energy negative and so (excepting the Lagrange case) have syzygies. If the brake time is t_0 then the brake orbit satisfies $x_i(t_0 + t) = x_i(t_0 - t)$ where $x_i(t)$ is the position of the ith mass at time t. It follows that the syzygies immediately preceding and immediately following the brake time are of the same type 1,2, or 3, (assuming these nearby syzygies are not collisions) and so the orbit has a stutter in a time interval containing the brake time. See Figure 1 for an example.

Example 1.2 As a special case of the previous class of examples, consider the isosceles three-body problem in which masses 1 and 2 are equal. Start the masses at rest in the configuration of nearly flat, obtuse isosceles triangle in which 3 is place slightly above the midpoint of the edge 12. Then 3 will oscillate across the 12 edge a number of times before 1 and 2 collide. Consequently in such a solution we will have a syzygy sequence $33333\ldots *$ where we put a $*$ to indicate collision. A slight off-center perturbation of the initial configuration will resolve the collision into a quick "Kepler event", yielding a non-collision solution with sequence $333333\ldots.312\ldots.$ The initial isosceles solution has zero angular momentum and negative energy, and we can arrange for the perturbed one to continue to have this same angular momentum and energy.

Example 1.3. Within the non-zero angular momentum planar three-body problem and its special limiting cases we have found three published instances of stutters, or more precisely, of reversals of the 12 angular momentum. (See more concerning the relation between reversals and stutters in the following paragraph). Chenciner and Llibre [4] establish the existence of solutions having infinitely many reversals within the restricted circular planar three-body problem. Their solutions lie within their "punctured KAM torii". Fejoz, in a series of two papers [6], [7] establishes the existence of solutions having infinitely many reversals in the full planar three-body problem. His solutions exist as a corollary of a thorough study of the secular system and a careful application of KAM methods. Finally, reversals are known to exist in the problem of two fixed centers.

The purpose of this paper is to establish the existence of another open family of stuttering orbits within the zero angular momentum problem. We find the existence of stutters (or angular momenum reversals) more surprising in the zero-angular momentum problem than in the non-zero angular momentum problem, in part due to their clearer connection to variational principles. Our methods and point of view are perturbational, and so in a certain sense close to those of [4], [6], [7]. However, our methods are more elementary in that we never invoke or even touch the KAM theorem.

For the orbits of our family the distance r between masses 1 and 2 remains relatively small over a long period of time while the distance of either 1 and 2 from the third mass remains very large. Masses 1 and 2, which we refer to as "the bound pair" will execute nearly Keplerian motions about their common center of mass. The angular momenta J_1, and hence the eccentricity e of the instantaneous Keplerian ellipses for the two close masses changes slowly, with J_1 passing through 0 (and e passing through 1) at which time the orientation of the ellipse reverse, and the masses traverse their instantaneous ellipses in the opposite directions. $J_1 = 0$ corresponds to the collinear collision-ejection orbit in the $1 - 2$ Kepler problem. If we think of 1 as the earth and 2 as the moon, then the moon's orbit gets skinnier and skinnier until the point where its sense of rotation about the earth instantaneously stops and the moon begins to go around in the opposite direction while the orbit widens out again! See Figures 1 and 2.

Our method here is perturbation-theoretic. Write r for the distance between the two close masses and ρ for the distance between their center of mass and the distant mass. Let r_M be a 'typical' maximum value of r over some time interval, and let ρ_0 be the initial value of ρ. Then

$$\epsilon = r_M/\rho_0 \tag{1.1}$$

will act as the small parameter for our problem. If H_1 denotes the energy of the bound pair then we can take

$$r_M = -\beta_1/H_1$$

where $\beta_1 = m_1 m_2$ is the Kepler constant occurring in H_1. Let θ be the angle between the Jacobi vectors, namely the vector ξ_1 which joins 2 to 1 and the vector ξ_2 which joins the 12 center of mass to 3. Let P_K be the period of the Kepler problem associated to H_1. (See Appendix 1 for formulae and precise definitions of the quantities just discussed.)

Theorem 1.1. *Consider the zero angular momentum three-body problem with negative total energy and any mass ratio. Let N be any positive integer. Then there exist two open families of solutions having stutters. One family has syzygy sequence $(12)^N 11 (21)^N$ and the other family has syzygy sequence $(21)^N 22 (12)^N$. The length of the time intervals over which this syzygy sequence is executed is $2NP_K + O(\epsilon)$ and there are no collisions during this interval. The family is characterized by initial conditions at a time t_* between the stutter syzygies 11 or 22. Among these conditions are that $r_M/\rho(t_*) < \epsilon$ where $\epsilon \to 0$ as $N \to \infty$, $H_1 < 0$ i , $J_1 = 0$, $\dot\rho = O(1)$ (as $\epsilon \to 0$), and $\cos(\theta)\sin(\theta) = O(1) \neq 0$. If $\cos(\theta)\sin(\theta) > 0$ at t_* then the stutter for the family is of type 11, while if $\cos(\theta)\sin(\theta) = O(1) < 0$ at t_* then the stutter is of type 22. The only zero of J_1 in this time interval is at t_* and it is a transverse zero.*

The proof of the theorem relies heavily on two equations for J_1:

$$J_1 = r^2 \left(1 + O\left(\frac{1}{\epsilon}\right)\right)\dot\theta$$

and

$$\dot J_1 = \frac{r^2}{\rho^3} F \cos(\theta)\sin(\theta) \qquad F = O(1) > 0$$

The first equation requires the total angular momentum to be zero. The second holds generally.

The main idea of the proof. In the motions we consider, the bound partners 1 and 2 travel rapidly along near-Keplerian ellipses, while body 3 is far away and slow moving. The effect of body 3 is to add a weak and slowly varying gravitational field to the Kepler problem 1-2. As a first approximation, freeze the position of 3, and approximate its gravitational field in a small disk including 1 and 2 by its linear part. Upon choosing coordinates so that 3 lies on the positive x-axis this additional force has the form $\varepsilon(x, 0)$. Thus the vector $\xi_1 = (x, y)$ connecting body 2 to 1 evolves, in our first approximation, according to

$$\ddot\xi_1 = -\xi_1/|\xi_1|^3 + \varepsilon(x, 0), \tag{1.2}$$

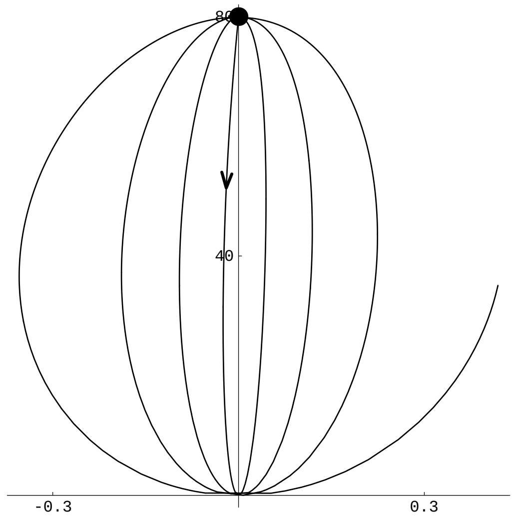

FIGURE 1. The curve traced by the short Jacobi vector for a brake orbit. The three bodies have masses $m_1 = 7, m_2 = 1$ and $m_3 = 13$ in units with the gravitational constant equal to 1. The initial positions at the brake time (indicated with the dot) are $(x_1, y_1) = (325, 10), (x_2, y_2) = (325, -70)$ and $(x_3, y_3) = (-200, 0)$ and the initial velocities are all zero.

A stutter corresponds to two consecutive crossings of the positive x–axis.

We explain stutters in this simpler system, and then outline our later steps in the proof of stutters in the full system. Figure 2 illustrates the motion $\xi_1(t) = re^{i\theta}$ with $\xi_1(0)$ in the first quadrant and with $\dot{\xi}_1(0) \parallel \xi_1(0)$. The key observation is that the negative tidal torque causes angular momentum $J = \xi_1 \wedge \dot{\xi}_1 = r^2\dot{\theta}$ to decrease monotonically in the first quadrant: indeed,

$$\frac{d}{dt}J = \frac{d}{dt}(\xi_1 \wedge \dot{\xi}_1) = \xi_1 \wedge \ddot{\xi}_1 \overset{(1.2)}{=} -\varepsilon xy < 0.$$

Thus for $t > 0$ we have $J(t) < 0$ and hence $\dot{\theta} = Jr^{-2} < 0$ for as long as z stays in the first quadrant. This implies that when z leaves the first quadrant[1], it does so by crossing the positive x–axis, as shown in Figure 2. By an identical argument z crosses the same semi-axis for some $t < 0$. Two consecutive crossings of the positive x–axis correspond to a stutter.

We now outline our approach to the problem without the simplifying assumptions. We address at the same time the problem of proving that stutters are preceded by many revolutions of the primaries, provided that the third body is far away and slow. As above, let ξ_1 be the vector for the bound system: the vector joining 1 and 2, while ξ_2 connects the $1-2$ center of mass to the distant 3rd body.

We will specify a set of initial data for the bodies 1 and 2 which yield bounded Keplerian ellipses with a fixed upper bound on the major semiaxis for the two-body problem unperturbed by the third, with the third body starting far away ($|\xi_1(0)| = \rho >> 1$) and slow ($|\dot{\xi}_1(0)| < \rho^{-1}$). We then show that with such initial data the two primaries move along slowly changing Keplerian ellipses for many

[1] and it is easily shown that it does

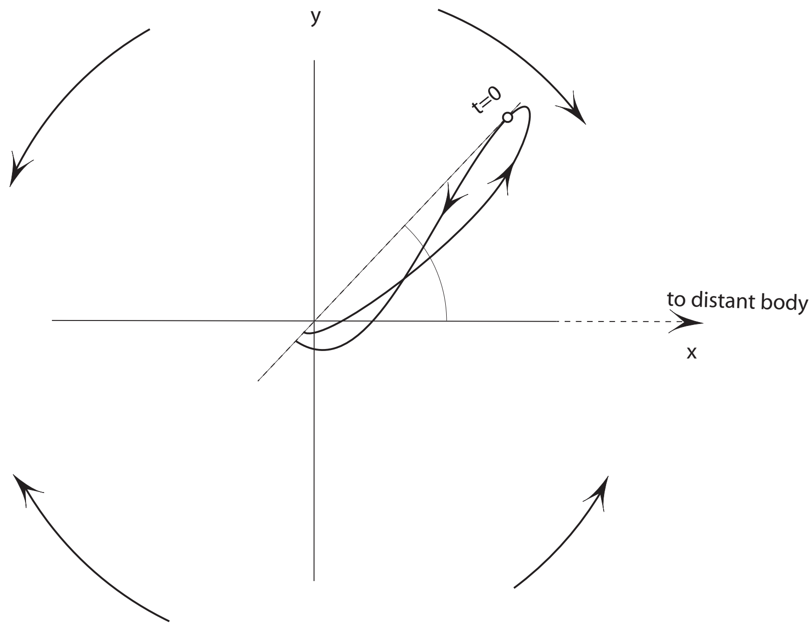

FIGURE 2. The evolution of the short Jacobi vector ξ_1 connecting the two bound masses. Arrows indicate the direction of the "tidal" torque.

revolutions if ρ is large. We will prove this in the body of the paper by carrying out the following steps.

i) We consider the perturbed Kepler problem

$$\ddot{\xi}_1 = -\xi_1/|\xi_1|^3 + f(\xi_1, t), \tag{1.3}$$

and estimate the slowness of change of the ellipses in terms of the norm of f.

ii) Considering a single body in the force field: $\ddot{\xi}_2 = -\xi_2/|\xi_1|^3 + f_2(\xi_2, t)$, with $|f_2(\xi_2, t)| <$ const. $|\xi_2|^{-1}$ for large $|\xi_2|$, we show that if the body starts with

$$|\xi_2(0)| \geq \rho, \quad |\dot{\xi}_2(0)| \leq \rho^{-1}, \tag{1.4}$$

then

$$|\xi_2(t)| \geq \frac{1}{2}\rho, \quad |\dot{\xi}_2(t)| \leq 2\rho^{-1} \tag{1.5}$$

for a long time: $0 \leq t \leq T(\rho) \to \infty$ as $\rho \to \infty$.

iii) We now consider our full problem with initial conditions such that the initial instantaneous major semiaxis of the two primaries (disregarding body 3) is $\leq r_M = O(1)$ and the third body satisfies (1.4). Our goal is to show that the two primaries will move in near-Keplerian ellipses with the third body staying far away **for a long time**: $-\tau(\rho) \leq t \leq \tau(\rho) \to \infty$ as $\rho \to \infty$. To that end let τ be the first time that either the major semiaxis of the Kepler ellipse exceeds $2r_M$, or body 3 fails (1.4). It is clear from the estimates mentioned that $\tau \to \infty$ as $\rho \to \infty$. Indeed, the semiaxis changes slowly since 1.5 implies that the Kepler perturbation H in (1.3) satisfies conditions of (i), and thus it takes a long time before the condition of semiaxis $\leq 2r_M$ to be violated. On the other hand, it also takes a long time for (1.5) to become violated, according to (ii).

2. **Motivation.** One of the primary motivations behind this article was to better understand the limits of action minimization for understanding the planar three-body problem. A host of new solutions to that problem have been constructed using the method of action minimization, one of the earliest being [5]. But none of these variational minimizers can have stutters. See the part of the proof of Lemma 7 going from the bottom of p. 896 to the top of p. 897 of [5]. We recall the construction there:

Proposition 2.1 (Reflection principle)**.** *Suppose that the planar three-body curve* $\gamma : [0, T] \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ *minimizes the action functional subject to the constraint of fixed end points, or to endpoints lying on fixed rotationally invariant subvariety. Then* γ *has no at most one syzygy on the open interval* $(0, T)$*, and in particular,* γ *will suffer no syzygies on this interval.*

Proof. We work on shape space, as in [5]. Let π be the projection from the standard configuration space to shape space. Suppose there were two (or more) interior stutters, so that at $t = a$ and $t = b$ the curve suffers syzygies. Suppose also that the curve is not everywhere collinear, as we do not count collinear intervals – continuous syzygies – as syzygies. Then $\pi \circ \gamma$ has crossed the collinear plane in shape space at two points. The curve γ must be a solution to Newton's equations away from collisions, and indeed there are no interior collisions by Marchal's theorem (see the exposition of [3]) so the curve is a solution to Newton's equations on $(0, T)$. It follows that both crossings of the collinear plane are transversal, since if either were tangential that initial condition would yield that the entire solution were collinear. Now the operation of reflecting a curve about the collinear plane in shape space preserves the action. Reflect the arc $\pi \circ \gamma([a, b])$, while keeping the rest of the γ the same. The resulting curve $\tilde{\gamma}$ still minimizes subject to the same constraint, and hence solves Newton's equations. This contradicts the uniqueness of solution to ODEs, (and also contradicts the fact that solutions must be analytic). Hence $\tilde{\gamma}$ must not exist, which means there could not have been the two interior syzygies in the first place. □

Remark 2.2. The reader might wonder if these reflection operations can really be implemented in inertial space. A reflection about any line implements reflection about the equator in shape space. Then reflect $\gamma([a, b])$ about the line defined by the configuration $\gamma(a)$. The resulting curve $\tilde{\gamma}$ will no longer satisfy $\gamma(b) = \tilde{\gamma}(b)$, but rather $\tilde{\gamma}(b) = R\gamma(b)$ where R is the reflection about the line of $\gamma(a)$. Reflect the remaining curve $R\gamma([b, T])$ about the line $R\gamma(b)$ to finish off the curve. The assumption on the boundary conditions over which γ minimizes guarantees that this new curve $\tilde{\gamma}$ continues to satisfy the boundary conditions.

In [13] the reflection principle plus the fact that the Jacobi-Maupertuis metric is negatively curved yields:

Proposition 2.3. *For the equal mass three-body problem with a* $1/r^2$ *attractive potential there are no bounded solutions with stutters.*

3. **Set-up. Key equations.** Let $x_i \in \mathbb{R}^2$ be the positions of the bodies and $m_i > 0$ their masses. We will express the dynamics in terms of the standard Jacobi vectors as discussed above:

$$\xi_1 = x_1 - x_2$$

and

$$\xi_2 = x_3 - (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$$

For the lengths of the Jacobi vectors we write

$$r = |\xi_1|, \qquad \rho = |\xi_2|.$$

Our solutions will remain in the region

$$\rho >> r$$

over the time interval of our analysis. We can write the total angular momentum J as

$$J = J_1 + J_2$$

where the J_i are the angular momenta for the individual Jacobi vectors: $J_i = \alpha_i \xi_i \wedge \dot{\xi}_i$ and

$$\alpha_1 = m_1 m_2 / M_{12} \tag{3.1}$$

$$\alpha_2 = m_3 M_{12} / M \tag{3.2}$$

where $M_{12} = m_1 + m_2$ is the mass of the 12 pair while $M = m_1 + m_2 + m_3$ is the total mass. J is conserved. The J_i are not. We can write the total energy as $H = H_1 + H_2 - g$ where the H_i are the effective Kepler energies for each Jacobi vector and $g = O(1/\rho^3)$ encodes the interaction between the Jacobi vectors. See Appendix 1 for explicit formulae. We write θ for the angle between the two Jacobi vectors and ϕ for the angle between the long Jacobi vector and a fixed inertial axis. Then:

$$\begin{aligned} J_1 &= \alpha_1 r^2 (\dot{\theta} + \dot{\phi}) \\ J_2 &= \alpha_2 \rho^2 \dot{\phi}, \end{aligned}$$

We can solve for $\dot{\phi}$ in terms of $\dot{\theta}$ and J:

$$\dot{\phi} = \frac{1}{I}\left(J - \alpha_1 r^2 \dot{\theta}\right)$$

where

$$I = \alpha_1 r^2 + \alpha_2 \rho^2$$

is the total moment of inertia. Plugging back in to the J_1 equation we find that

$$J_1 = \alpha_1 r^2 \left(1 - \alpha_1 \frac{r^2}{I}\right)\dot{\theta} + \alpha_1 \frac{r^2}{I} J.$$

In the special case that $J = 0$ we get a linear, rather than affine, relation between J_1 and $\dot{\theta}$:

$$\begin{aligned} J_1 &= \alpha_1 r^2 \left(1 - \alpha_1 \frac{r^2}{I}\right)\dot{\theta} \\ &= \alpha_1 r^2 \left(1 - O\left(\frac{1}{\epsilon^2}\right)\right)\dot{\theta} \end{aligned} \tag{3.3}$$

This linear relation between J_1 and $\dot{\theta}$, valid only when $J = 0$, will make our further analysis simpler and is the main reason we made the hypothesis $J = 0$ in the statement of the theorem.

3.1. **Key equation.** Our analysis rests on the equation

$$\frac{d}{dt}J_1 = \frac{r^2}{\rho^3}F\cos\theta\sin\theta \tag{3.4}$$

with

$$F = \frac{3m_1m_2m_3}{m_1+m_2} + O\left(\frac{r}{\rho}\right) > 0 \tag{3.5}$$

This equation is derived in Appendix 1. The equation is illustrated by Figure 2. The equation is valid regardless of whether or not $J = 0$. The torque, being the right hand side of eq. (3.4) is zero if and only if $\theta = 0, \pi/2, \pi, 3\pi/2$. If $\theta = 0$ or π then the collinear solutions provide examples for which $J_1 \equiv 0$. If $\theta = \pi/2$ or $3\pi/2$ and $m_1 = m_2$ then the isosceles solutions provide examples in which $J_1 \equiv 0$.

4. **Proof of the main theorem.** We use the notation in and around the statement of the theorem, as further detailed in the previous section. The angle $\theta(t)$ between $\xi_1(t)$ and $\xi_2(t)$ is well-defined (modulo 2π) and continuous as long as neither vector is zero. To insure that neither vector is zero we will prove that

$$0 < r < \rho \tag{4.1}$$

over an interval $[t_* - T, t_* + T]$ with $T = NP_K + O(\epsilon)$ as per the theorem. When $\theta = 0$ we have an eclipse of type 1 and when $\theta = \pi$ we have an eclipse of type 2. Over any subinterval on which θ varies monotonically, the eclipses of type 1 and 2 must alternate.

By translating time we may suppose that the time t_* of the theorem is $t_* = 0$. Let us suppose that $\dot\theta(0) = 0, \theta(0) \neq 0, \theta(0) \neq \pi(\mod 2\pi)$, and that $\dot\theta < 0$ for $t < 0$ and $\dot\theta > 0$ for $t > 0$, and $t \in [-T, T]$. For the sake of argument, suppose $-\pi < \theta(0) < 0$. Then, since θ is monotonically decreasing for $t < 0$ and monotonically increasing for $t > 0$ the syzygies on either side of $t = 0$ are both of type 1. We have our stutter! And we have long intervals on either side of $t = 0$ over which the syzygies are alternating of type 12.

The argument presented presupposed the existence of syzygies in $[-T, 0]$ and in $[0, T]$. **The bulk of the remainder of the paper is used to establish the existence of these syzygies, roughly $2T/P_K$ of them in each half-interval.** We will show that on any interval not containing zero whose length is approximately P_K that the θ varies by approximately 2π. It follows that ξ_1 winds approximately T/P_K times around the origin on either side of $t = 0$, and in so doing the solution sweeps out a 'tight binary' syzygy sequence 121212.... with roughly N of the 12's.

We remark that the inequality 4.1 excludes syzygies of type 3 from occurring. Indeed, if 3 does pass between 1 and 2 on the line joining them, then its distance ρ to the 12 center of mass is shorter than r, the distance between 1 and 2.

To summarize, a solution will execute a syzygy sequence of the type described in the theorem over a time interval $[-T, T]$ with $T = NP_K + O(\epsilon)$ provided that for t in this interval we have

- Fact 1. $r(t) < \rho(t)$.
- Fact 2. $0 < r(t)$
- Fact 3. $\dot\theta$ has a unique transverse zero at a time $t = 0$
- Fact 4. The angle $\theta(t)$ varies by $2\pi + O(\epsilon)$ over each subinterval $[a, b] \subset [-T, T]$ of size $b - a \cong P_K$ not containing 0
- Fact 5. $\theta(0) \neq 0, \pi$

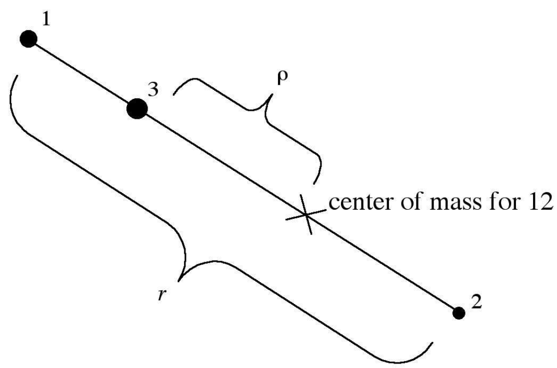

FIGURE 3. A syzygy with 3 in the middle requires $r > \rho$ at that moment.

The bulk of the remainder of the paper is devoted to establishing these facts.

Instead of Fact 1 we will use energy bounds to prove the stronger:

• *Fact* $1'$. $r/\rho < 2\epsilon$. Fact $1'$ will be proved as Bound 6 within Appendix 3.

We will see momentarily that Fact 2 follows from Fact 3.

Since $J_1 = (pos.)\dot{\theta}$, when the total angular momentum J is zero, Fact 3 is equivalent to the analogous fact concerning J_1:

• *Fact* $3'$. J_1 has a unique transverse zero at a time 0 in the interval

Fact 4 will follow from the "Kepler approximation" discussed in the next section.

Fact 5 is the assumption $\cos(\theta(t_*))\sin(\theta(t_*)) \neq 0$ made on the initial data in the hypothesis of the theorem .

To see that Fact 2 follows from Fact $3'$, we recall a well-known fact, reproved here in Appendix 2, that at binary collision $r = 0$ and $\rho \neq 0$, we must have that $J_1 = 0$. Since $J_1 \neq 0$ for $t \neq 0$ we cannot have $r(t) = 0$ for $t \neq 0$ in the interval. And $r(0) \neq 0$ by assumption.

It remains to verify Facts $1'$, $3'$, and 4. The most difficult of these facts to verify is Fact $3'$. For its verifications we need:

4.1. **Kepler approximation; Levi-Civita transformation.** The short Jacobi vector $\xi_1(t) = x_1 - x_2$ satisfies a differential equation of the form

$$\alpha_1 \ddot{\xi}_1 = -\beta_1 \frac{\xi_1}{r^3} + f \tag{4.2}$$

where α_1, β_1 are positive constants depending on the masses and where

$$f = O(1/\rho^3)$$

depends on ξ_2 as well as ξ_1, and represents the forces of body 3 on 1 and 2. The derivation of (4.2) and the precise equation is found in Appendix 1. The Kepler approximation is the act of setting $f = 0$ above. We say "the Kepler approximation is valid" over a time interval if the real motion of ξ_1 with f present stays close to the solution of the Kepler approximation equation over that interval. We define the "instantaneous Kepler orbit" of ξ_1 at time t_0 to be the solution to the Kepler approximation with initial conditions $\xi_1(t_0), \dot{\xi}_1(t_0)$. Chazy [2], and many others have established the validity of the approximation under various different regimes. Due to the singularity at collision $r = 0$, Chazy explicitly excluded our case in which the instantaneous Kepler orbit has zero eccentricity. To allow for our case we need to regularize binary collision. We will use the Levi-Civita regularization [8], also called the Bohlin transformation. See Robinson, [16] for a particularly nice version of this regularization useful for our domain of initial conditions.

The Levi-Civita transformation is:

$$z^2 = \xi_1 \qquad\qquad d\tau = dt/|\xi_1(t)| \tag{4.3}$$

a simultaneous transformation of both the dependent (ξ_1) and independent (t) variables. In Appendix 4 we show that under this change of variables eq. (4.2) becomes a perturbed harmonic oscillator:

$$z'' = \frac{H_1}{2} z - \frac{1}{2} z\bar{z}^2 f. \tag{4.4}$$

where the $'$ denotes differentiation with respect to τ, where

$$H_1 = \frac{\alpha_1}{2} \left|\dot{\xi}_1\right|^2 - \frac{\beta_1}{r}$$

is the (instantaneous) Kepler energy of ξ_1, and where f is the perturbation term in eq (4.2). Let $z_0(\tau)$ be the Levi-Civita transform associated to the Kepler approximation, so that $z_0^2 = \xi_0$, $d\tau = dt/|\xi_0(t)|$, and $\xi_0(t)$ is the instantaneous Kepler orbit at $t = 0$. Then the same computation yields an exact harmonic oscillator

$$z_0'' = \frac{H_1(0)}{2} z_0 \tag{4.5}$$

Let P be the half-period for $z_0(\tau)$, which is the Levi-Civita transform of P_K (so $P_K = \int_0^P |z_0(\tau)|^2 d\tau$; see Appendix 5 .) In Appendix 5 we prove that there exist constants c and ϵ_0 such for $\epsilon < \epsilon_0$ and all $|\tau| < NP$ we have

$$|z(\tau) - z_0(\tau)| < c\epsilon \tag{4.6}$$

4.2. **Validity of Fact** $3'$**.** We use inequality (4.6) to prove the validity of Fact $3'$. We assume that the term $\cos(\theta_0)\sin(\theta_0)$ appearing in the key eq. (3.4) is positive, and $O(1)$. We work in the τ variables of the previous section. We will prove that, provided ϵ is small enough that there is a positive constant δ such that

$$J_1(\tau) > (\epsilon^3\delta)\tau \text{ for } \tau > 0 \text{ while } J_1(\tau) < (\epsilon^3\delta)\tau \text{ for } \tau < 0 \tag{4.7}$$

for τ varying over the image of our interval $[-T, T]$, (where $T = NP$), under the Levi-Civita transform $t \mapsto \tau = \tau(\xi_1(t); t)$. Inequality 4.7 plus the monotonicity of the map $t \mapsto \tau$ clearly implies Fact $3'$. If instead we assume that $\cos(\theta_0)\sin(\theta_0) < 0$ and is $O(1)$, then a nearly identical argument will yield positive δ such that $J_1(\tau) < -(\epsilon^3\delta)\tau$ for $\tau > 0$ while $J_1(\tau) > -(\epsilon^3\delta)\tau$ for $\tau < 0$.

Since

$$\frac{d}{d\tau} J_1 = \frac{dt}{d\tau}\frac{d}{dt} J_1 = r\frac{d}{dt} J_1$$

we have that

$$J_1'(\tau) = \frac{r^3}{\rho^3} F \sin(\theta)\cos(\theta).$$

Integrating, and using $J_1(0) = 0$ we have that

$$J_1(\tau) = \int_0^\tau \frac{r^3}{\rho^3} F \sin(\theta)\cos(\theta) d\tau.$$

(All variables are to be expressed as functions of τ.)

The argument proceeds, roughly speaking, by arguing that inside the integral the function F/ρ^3 is nearly constant and equal to $F_0\epsilon_0^3$, while $r^3\sin(\theta)\cos(\theta)$ can be replaced, with little error, by the corresponding function we would get by using $z_0(\tau)$ instead of $z(\tau)$, the latter being the Levi-Civita transform of the exact solution. Let $J_1^0(\tau)$ be the result of making these two replacements. To write down $J_1^0(\tau)$ more explicitly recall that $\xi = z^2$ so that $r = |z|^2$ under the Levi-Civita transformation. Then $r^3\sin(\theta)\cos(\theta) = rxy$ where $x = r\cos(\theta), y = r\sin\theta$ are the components of ξ. Writing $z = u + iv$. we obtain

$$r^3\sin(\theta)\cos(\theta) = 2(u^2+v^2)(u^2-v^2)uv := Q(z). \tag{4.8}$$

Substituting in $z = z_0(\tau)$ to $Q(z)$, remembering to replace F/ρ^3 by $F_0\epsilon_0^3$, we obtain $J_1^0(\tau) = F_0\epsilon^3\int_0^\tau Q(z_0(s))ds$.

We will show that there are constants δ_1 and C, independent of ϵ (depending only on H_1 and the bounds on $\dot\rho(0)$) such that

$$J_1^0(\tau) > \epsilon^3\delta_1\tau \;\text{ for } \tau > 0 \;\text{ and }\; J_1^0(\tau) < \epsilon^3\delta_1\tau \;\text{ for } \tau < 0 \tag{4.9}$$

while over the intervals $[-NP, NP]$ we have

$$|J_1(\tau)) - J_1^0(\tau)| < C\epsilon^4((\tau)^2 + \tau). \tag{4.10}$$

The two bounds (4.9) and (4.10) together yield

$$J_1(\tau) > \epsilon^3\delta_1\tau - \epsilon^4((C\tau^2 + C\tau) = \epsilon^3(\delta_1 - \epsilon C - \epsilon C\tau)\tau.$$

By choosing ϵ so small that $(\delta_1 - \epsilon C - \epsilon CT) > \frac{1}{2}\delta_1 := \delta$ we guarantee that for $0 < \tau < T = NP$, $J_1(\tau) > \epsilon^3\delta\tau$ which is the desired bound (4.7).

4.3. **Proof of the Bound (4.9).** The proof of this bound depends only on the fact that $J_1^0(\tau)$ is the indefinite integral of the continuous *non-negative periodic* function $g(s) = Q(z_0(s))$ and that $g(0) > 0$. The periodicity of g follows from that of the Kepler solution. Its non-negativity is a result of the fact that the angle θ of a collinear Kepler solution is constant, so that $\cos(\theta)\sin(\theta) > 0$ everywhere along the solution. The fact that $g(0) > 0$ is a consequence of the choice of non-collision initial condition: $r(0) > 0$. The zeros of g correspond to the collisions.

Lemma 4.1. *Let $g(s)$ be a continuous non-negative periodic function of period P. Suppose that $g(0) > 0$ and set $J(t) = \int_0^t g(s)ds$ (J plays the role of $J_1^0(\tau)$ in the inequality.) Then there exists a positive constant k such that for all $t > 0$ we have $J(t) > kt$.*

Proof. Let c be the average of g over one period. Note that $c = J(P)/P$. We will show that $k = \inf_{0<s<P}(J(s)/s)$ is positive, and is the constant of the lemma.

Step 1. $J(t+P) = J(t) + cP$

Proof of Step 1: Subtracting the average c from the integrand $g(s)$ to obtain the continuous periodic function $\phi(t) = \int_0^t (g(s)-c)ds$ of period P. Since $J(t) = ct+\phi(t)$ and ϕ is periodic we have $J(t+P) = c(t+P) + \phi(t) = cP + J(t)$.

Step 2. Since $g \geq 0$ and g is not identically zero we have $c > 0$ and $J(s) > 0$ for $s > 0$.

Step 3. By the fundamental theorem of calculus the function $J(s)/s$ has limit $g(0)$ as $s \to 0$. Thus $J(s)/s$ is a continuous function on $[0, P]$. Since $g(0) > 0$ and $J(s) > 0$ on the interval the function $J(s)/s$ is an everywhere positive function on the interval. By continuity its infimum over the interval, which we called k above, is realized and is a positive number. We now have $J(t) \geq kt$ over the interval $[0, P]$.

Step 4. Use Steps 1 and 3 and recall that $c = g(P)/P$ so that $c \geq k$. We get, for $0 < t < P$: $J(t+P) = cP + J(t) \geq kP + kt = k(P+t)$. We now have $J(t) \geq kt$ for $0 \leq t \leq 2P$ By induction, $J(t+nP) \geq k(t+nP)$ for all t, $0 < t < P$ and for all positive integers n. Thus $J(t) \geq kt$ for all positive t. $\square$

4.4. **Proof of the Bound (4.10).**

Proof. Consider the polynomial $Q(z) = Q(u,v)$ occurring in (4.8). Since $z_0(\tau)$ is bounded, Q is Lipschitz (being analytic), and $|z(\tau) - z_0(\tau)| < C\epsilon$ (inequality (4.6)) for $|\tau| < T = NP$ we have that

$$|Q(z(\tau)) - Q(z_0(\tau))| < L\epsilon$$

for some constant L and for $|\tau| < T = NP$.

It follows that

$$\left| \int_0^\tau \frac{F_0}{\rho_0^3} Q(z(s))ds - \int_0^\tau \frac{F_0}{\rho_0^3} Q(z_0(s))ds \right| < |F_0|\epsilon^3 L\epsilon\tau = C\epsilon^4\tau$$

Next, write

$$\begin{aligned} J_1(\tau) &= \int_0^\tau \frac{F(s)}{\rho(s)^3} Q(z(s))ds \\ &= \int_0^\tau \frac{F_0}{\rho_0^3} Q(z(s))ds + \int_0^\tau \left(\frac{F(s)}{\rho(s)^3} - \frac{F_0}{\rho_0^3} \right) Q(z(s))ds \end{aligned} \tag{4.11}$$

so that, for the desired result, it suffices to bound the last integral. We achieve this through the two bounds:

$$|F(\tau) - F_0| \leq C\epsilon \tag{4.12}$$

valid over the time interval in question, and

$$\left| \frac{1}{\rho(s)^3} - \frac{1}{\rho_0^3} \right| < 2C\epsilon^4 s \tag{4.13}$$

$\square$

4.5. **Proof of Bound (4.12).**

Proof. In Appendix 1, eq. (6.10) (see also the equations immediately preceding it, notably the one involving $r\rho\gamma_2 \sin(\theta)$) we find that $F = F_0 + O(r/\rho)$. In Appendix 3, Bounds 1 and 6 we show that r and $1/\rho$ are bound over the interval in question, so that $r/\rho < C\epsilon$. $\square$

4.6. **Proof of Bound (4.13).**

Proof.

$$\begin{aligned} \left| \frac{1}{\rho(s)^3} - \frac{1}{\rho_0^3} \right| &= \left| \int_0^s \frac{d}{dt} \frac{1}{\rho^3(t)} dt \right| \\ &= \left| - \int_0^s \frac{3\nu(t)}{\rho^4(t)} dt \right| \\ &\leq 3\bar{\nu}(2\epsilon)^4 |s| \\ &\leq 2C\epsilon^4 |\tau| \end{aligned}$$

where in the next-to-last line we used $\dot{\rho} := \nu(t)$ and the speed bound

$$|\nu(t)| \leq \bar{\nu} \tag{4.14}$$

and

$$\frac{1}{\rho} < 2\epsilon \tag{4.15}$$

over the interval in question. The bound (4.15) is the point of Bound 6 in Appendix 3. The bound (4.14) is not stated explicitly in the Appendices, but follows directly from the Sandwich Lemma crucial to Bound 6 in Appendix 4. (The constant C of $2C\tau$ is then $24\bar{\nu}$.) $\square$

4.7. **Finishing the Bound (4.10).**

Proof. Writing

$$\frac{F}{\rho^3} - \frac{F_0}{\rho_0^3} = F\left(\frac{1}{\rho(t)^3} - \frac{1}{\rho_0^3}\right) + \frac{F - F_0}{\rho_0^3}$$

we see that

$$\left|\frac{F(s)}{\rho(s)^3} - \frac{F_0}{\rho_0(s)^3}\right| < 2C\epsilon^4|s| + C\epsilon^4$$

Integrating as in eq (4.11) and using $|Q(z)| < C$ over the disc within which $z(\tau)$ lies, yields the desired bound on the second integral in eq (4.11), so that when added to the bound already established on the first integral there, yields the final result, (4.10). $\square$

4.8. **Verification of Fact 4.** The collinear Kepler approximant $\xi_0 = z_0^2$ oscillates periodically through the origin. At its farthest reach it lies a distance $r = r_M = |H_1(0)|/\beta_1 = O(1)$ from the origin. Its oscillation period is

$$P_K = \frac{1}{\beta_1\sqrt{\alpha_1}}\left(2|H_1(0)|\right)^{3/2}$$

It follows from these facts, and the validity of the Kepler approximation (Appendix 5) that the radial distance $r(t)$ for the short Kepler vector ξ_1 of the true motion varies between consecutive local maxima at $r_M + O(\epsilon)$ and that the time between these maxima is equal to $P + O(\epsilon)$. We choose a, b to be the times corresponding to two such local maxima, and write $b - a = T$, and note $T = P + O(\epsilon)$. We claim that if the interval $[a, b]$ does not contain zero (0 being the time at which J_1 switches sign) then the curve $\xi_1(t)$ must wind nearly one full time around the origin between these maxima.

Proof. To prove our claim we argue by contradiction. Let $\xi_1(a)$ and $\xi_1(b)$ be the points on the curve corresponding to the successive maxima. By the validity of the Kepler approximation $\xi_1(a), \xi_1(b)$ are nearly equal, as are their derivatives $v_1(a), v_1(b)$. These two facts follow from the fact (Appendix 5) that in the τ parameterization, z_0 and z are C^1-close, where $z(\tau)$ is, as above, the Levi-Civita transform of $\xi_1(t)$. (The $d\tau/dt = 1/r$ multiplicative factor relating the τ and t velocities at approximate aphelia $t = a, b$ is $O(1)$, so the velocities $v_1 = \dot{\xi}_1(t)$ at $t = a, b$, must be nearly equal. This near equality need not hold at successive approximate perihelia – minima for r – since these minima are $O(\epsilon)$ so that the multiplicative factor $d\tau/dt$ is $O(1/\epsilon)$.) Since $J_1 \neq 0$ in the given interval, there is no collision. (Appendix 2.) Also θ is monotone over the interval. (Equation (3.3).) It follows that if the curve does not wind around the origin then it must be contained within the small sector bounded by the rays from 0 to $\xi_1(a)$ and from 0 to $\xi_1(b)$ and interior to the circle $r = r_M + O(\epsilon)$. See Figure 4. We will show that this contradicts the facts of the

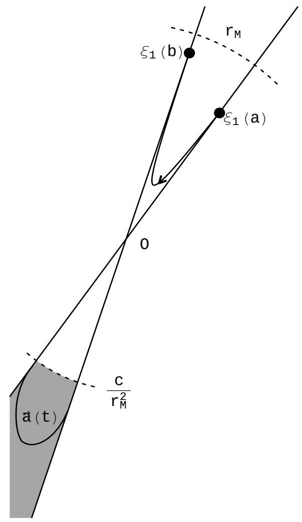

FIGURE 4. Convex hull argument for accelerations

acceleration, namely that it points primarily towards the origin. On the one hand:

$$|v_1(b) - v_1(a)| = O(\epsilon) \tag{4.16}$$

while on the other hand

$$|v_1(b) - v_1(a)| = \int \vec{a}(s)ds \tag{4.17}$$

where $\vec{a} = dv_1/dt = \ddot{\xi}_1$ is the acceleration. From the differential equation $\vec{a}(t) = -c\xi_1(t)/r(t)^3 + O(\epsilon)$ (see (4.2), and Appendix 1 and 3), and the restrictions on ξ_1, we see that the acceleration $\vec{a}(t)$ lies in a region which is an $O(\epsilon)$ perturbation of the region bounded by the negative of our sector, i.e. bounded by the rays from 0 to $-\xi_1(a)$ and from 0 to $-\xi_1(b)$ and *exterior* to the circle $c/r_M^2 + O(\epsilon) = |\vec{a}|$. The convex hull of this region is a thin wedge shaped region which is an order ϵ perturbation of the 'chopped' sector which is bounded by the same two rays and the chord joining where they intersect the circle. Now every point $\vec{a}$ in this convex region satisfies $\vec{a} \cdot \vec{n} < -c_1/r_M^2$ where $\vec{n}$ is a unit vector pointing in the direction of $\xi_1(a)$ and where $c_1 = c + O(\epsilon)$ is a constant. (The constant $c = \beta_1/\alpha_1$ is $O(1)$.) The average of any parameterized curve lying in a convex region again lies in that region. It follows that

$$\vec{n} \cdot \frac{1}{T}\int_0^T \vec{a}(s)ds < -\frac{c}{r_M^2}$$

Then $|\vec{n} \cdot (v_1(b) - v_1(a))| > Tc/r_M^2$, contradicting (4.16). ☐

We have now proved the validity of the five facts. As described above, the five facts together imply the main theorem.

5. **Open questions.**

i) Do there exist stuttering orbits of the type described in the theorem which tend towards circularity (eccentricity 0) as we evolve away from the $J_1 = 0$ instant (at which the eccentricity is 1)?
ii) Do there exist periodic stuttering orbits?

iii) Do there exist periodic non-collision brake orbits?

iv) Given any finite sequence $N_1, N_2, \ldots, N_k$ of large positive integers does there exist a solution whose syzygy sequence is $\alpha_1 s_1 \alpha_2 \ldots s_{k-1}\alpha_k$ where the s_i are stutters of length 2: either 11 or 22 and the α_i represent 'tight binary sequences: ..121212.... of length N_i, arranged so that the only stutters are the s_i? (Thus, if $s_1 = 11$ then α_1 ends in a 2 while α_2 begins with a 2.)

v) Is there a quantitative relation between the existence of stutters and conjugacy points for either the Lagrangian action, or the Jacobi-Maupertuis length?

vi) Can one use the methods of Fejoz [6], [7] or Chenciner-Llibre [4] to establish existence of stuttering solutions with zero angular momentum?

6. **The Appendices.** In the first five of the following six appendices we derive bounds used in the proof of the theorem. The sixth Appendix presents an example crucial to the formulation of our main theorem.

Appendix 1. Equations of motion in Jacobi variables. The key equation.

Appendix 2. $r = 0 \implies J_1 = 0$.

Appendix 3. Bounds on H_1, ρ, $\dot\rho$, the time interval, etc.

Appendix 4. The Levi-Civita transformation. A perturbed harmonic oscillator.

Appendix 5: An oscillator Gronwall inequality; validity of the Kepler approximation. A Gronwall inequality. Validity of the Kepler approximation.

Appendix 6: A perturbed harmonic oscillator which never winds around the origin.

6.1. **Appendix 1: Equations and bounds; the key equation.** As in Section 3.1 denote the positions of the masses as x_1, x_2 and x_3. Associated to the partition $12;3$ of the masses we have the Jacobi vectors and their lengths: :

$$\begin{array}{rclcrcl} \xi_1 &=& x_1 - x_2 & \qquad & |\xi_1| &=& r \\ \xi_2 &=& x_3 - x_{cm}^{12} & & |\xi_2| &=& \rho \end{array}$$

where

$$\begin{array}{rcl} x_{cm}^{12} &:=& (m_1 x_1 + m_2 x_2)/(m_1 + m_2) \\ &:=& \mu_1 x_1 + \mu_2 x_2 \end{array}$$

is the 12 center of mass and we have set

$$\mu_1 = m_1/(m_1 + m_2) \qquad \mu_2 = m_2/(m_1 + m_2).$$

We must express the potential in terms of the Jacobi vectors, and so we need formulae for the mutual distances $r_{ij} = |x_i - x_j|$ in terms of ξ_1, ξ_2. We have $r_{12} = r = |\xi_1|$. To express the other two distances use: $x_3 - x_1 = \xi_2 - \mu_2\xi_1$ and $x_3 - x_2 = \xi_2 + \mu_1\xi_1$ to obtain:

$$r_{13} = \|\xi_2 - \mu_2\xi_1\| \qquad r_{23} = \|\xi_2 + \mu_1\xi_1\| \tag{6.1}$$

Note $r_{13}, r_{23} = \rho(1 + O(r/\rho))$.

Newton's equations are $\alpha_i\ddot{\xi}_i = -\partial V/\partial\xi_i$ for $i = 1, 2$, where

$$-V = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$$

The total energy is $H = K/2 + V$ where $K = \Sigma m_i|\dot x_i|^2$ is twice the kinetic energy. In Jacobi coordinates, with the center of mass at the origin, we have $K = \alpha_1|\dot\xi_1|^2 + \alpha_2|\dot\xi_2|^2$, where

$$\begin{array}{rclcrcl} \alpha_1 &=& \beta_1/(m_1 + m_2) & \qquad & \beta_1 &=& m_1 m_2 \\ \alpha_2 &=& \beta_2/(m_1 + m_2 + m_3) & & \beta_2 &=& (m_1 + m_2) m_3 \end{array}$$

The total energy H can be expressed as the sum of two Kepler energies H_1 and H_2 and a perturbation term g which goes to zero with r/ρ:

$$H = H_1 + H_2 - g \tag{6.2}$$

with

$$H_1 = \frac{1}{2}\alpha_1 \|\dot{\xi}_1\|^2 - \beta_1/r \tag{6.3}$$

$$H_2 = \frac{1}{2}\alpha_2 \|\dot{\xi}_2\|^2 - \beta_2/\rho \tag{6.4}$$

The coupling term g is given by

$$g \quad = \quad \frac{\beta_2}{\|\xi_2\|} - \frac{m_1 m_3}{\|\xi_2 - \mu_2 \xi_1\|} - \frac{m_2 m_3}{\|\xi_2 + \mu_1 \xi_1\|}$$

It follows that Newton's equations can also be written

$$\alpha_i \ddot{\xi}_i = -\frac{\beta_i \xi_i}{|\xi_i|^3} + g_{\xi_i} \tag{6.5}$$

where the subscript g_{ξ_i} denotes the gradient of g with respect to ξ_i. The total energy is constant along solutions, as is the total angular momentum

$$J = J_1 + J_2$$

with

$$J_i = \alpha_i \xi_i \wedge \dot{\xi}_i.$$

In order to derive the key eq. (3.4) for the evolution of J_1 it is useful to write the equation of $\ddot{\xi}_1$ another way:

$$\alpha_1 \ddot{\xi}_1 = \gamma_1 \xi_1 + \gamma_2 \xi_2 \tag{6.6}$$

where the scalar functions γ_1, β are given by

$$\gamma_1 = -\frac{m_1 m_2}{r^3} - m_3 \left\{ \frac{\mu_2^2 m_1}{r_{13}^3} + \frac{\mu_1^2 m_2}{r_{23}^3} \right\} \tag{6.7}$$

while

$$\gamma_2 = \frac{m_3 m_1 m_2}{m_1 + m_2} \left\{ \frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} \right\} \tag{6.8}$$

Then

$$\begin{aligned} \frac{d}{dt} J_1 &= \xi_1 \wedge (\alpha_1 \ddot{\xi}_1) \\ &= \gamma_2 \xi_1 \wedge \xi_2 \\ &= r\rho\gamma_2 \sin(\theta) \end{aligned}$$

where we have used $\xi_1 \wedge \xi_2 = r\rho \sin(\theta)$.

As $r/\rho \to 0$ the equations (6.5) limit to two decoupled Kepler problems, one for ξ_1, the other for ξ_2. We will need some information on the asymptotics of this decoupling. To get the asymptotics of γ_2, g, and the g_{ξ_i} one can use the Legendre polynomials P_j. The P_j can be defined in terms of two arbitrary vectors ξ, q in a Euclidean space. Set $\epsilon = |q|/|\xi|$ and $\xi \cdot q = |\xi||q|\cos(\psi)$. Then:

$$\begin{aligned} \frac{1}{\|\xi - q\|} &= \frac{1}{\|\xi\|} \left\{ 1 + \epsilon \cos(\psi) + \epsilon^2 P_2(\cos(\psi) + \ldots + \epsilon^j P_j(\cos(\psi) + \ldots \right\} \\ &= \frac{1}{\|\xi\|} + \frac{P_1(\xi \cdot q)}{\|\xi\|^3} + \frac{P_2(\xi \cdot q)}{\|\xi\|^5} + \ldots + \frac{P_j(\xi \cdot q)}{\|\xi\|^{2j+1}} + \ldots \end{aligned}$$

We note that $P_1(x) = x$. The first few terms of the Legendre polynomials and some algebra yields that

$$r\rho\gamma_2 \sin\theta = \left(\frac{m_3 m_1 m_2}{m_1 + m_2}\right) \frac{3r^2}{\rho^3} \cos\theta \sin\theta + O\left(\frac{r^3}{\rho^4}\right) \sin(\theta)\cos(\theta)$$

It follows that

$$\frac{d}{dt} J_1 = \frac{r^2}{\rho^3} F \cos\theta \sin\theta \tag{6.9}$$

with

$$F = \frac{3 m_3 m_1 m_2}{m_1 + m_2} + O\left(\frac{r}{\rho}\right) \tag{6.10}$$

which is the promised key equation (3.4) of Section 2.1.

For later use we record here estimates for g and its two gradients. We compute

$$\begin{aligned} g &= \frac{m_3}{\rho} \left\{ (m_1 \mu_2^2 + m_2 \mu_1^2) \left(\frac{r}{\rho}\right)^2 P_2(\cos\psi) + O\left(\frac{r}{\rho}^3\right) \right\} \\ &= \frac{(m_1 \mu_2^2 + m_2 \mu_1^2) P_2(\xi_1 \cdot \xi_2)}{\rho^5} + O\left(\frac{r^3}{\rho^4}\right) \end{aligned}$$

so that

$$|g| \leq C r^2/\rho^3. \tag{6.11}$$

For the gradients we compute the estimates:

$$g_{\xi_1} = O(r/\rho^3) = O(1/\rho^3) \tag{6.12}$$

and

$$g_{\xi_2} = O(r^2/\rho^4) = O(1/\rho^4) \tag{6.13}$$

Note: In going from $O(r^m/\rho^k)$ to $O(1/\rho^k)$ we use the fact, valid under the conditions of theorem 1.1, that r can be uniformly bounded. See eq. (6.17) below.

6.2. **Appendix 2: $r = 0 \implies J_1 = 0$.** We prove that if, in the course of a solution, we have a binary collision of 1 and 2 (so $r = 0$), then $J_1 = 0$ at collision.

Indeed, the far energy H_2 (see eq. (6.4)), and the perturbation term g are uniformly bounded and continuous during the collision. Conservation of the total energy then implies that H_1 is uniformly continuous and bounded in a neighborhood of the collision time so that $r^2 H_1 \to 0$ at collision. Now $H_1 = \frac{\alpha_1}{2}\left(\dot{r}^2 + \frac{J_1^2}{r^2}\right) - \frac{\beta_1}{r}$ so that

$$r^2 H_1 = \frac{\alpha_1}{2}\left(r^2 \dot{r}^2 + J_1^2\right) - \beta_1 r$$

It follows that $r^2\dot{r}^2$ and J_1^2 go separately to zero as we approach the time of collision.

6.3. **Appendix 3: Energy, distance, and time bounds.** The main purpose of this Appendix is to prove Fact 1′ of Section 4. This fact asserts that there is a constant c (of $O(1)$ as $\epsilon \to 0$) such that for any solution satisfying the initial condition bounds at $t = 0$ we have that $r_M/\rho(t) < 2\epsilon$ is true over the interval $|t| < c/\epsilon$. (Recall $\epsilon = r_M/\rho(0)$.) A secondary purpose of this Appendix is to prove that the bound energy H_1 varies by no more than $O(\epsilon^2)$ over this same interval.

Along the way we prove a number of auxiliary estimates, some needed for Appendix 4 and 5. Most of these other bounds are well-known and we expect that most if not all can be found in some form in the text of Marchal [10], or in Levi-Civita's treatise [8], but have found it easier and more reliable to derive them.

We divide the bounds into 'instantaneous bounds' which are algebraic in nature, and 'integral bounds' which involve an integration.

Instantaneous bounds

- Bound 1. On H_1, the bound energy
- Bound 2.On r, the bound distance
- Bound 3. On J_1 - the bound angular momenta
- Bound 4. On the far transverse velocity

Integral bounds

- Bound 5. An approximating Kepler equation for ρ
- Bound 6. On ρ; Fact $1'$.
- Bound 7. On the variation of H_1.

Throughout this Appendix, c, c_i, C_i, k and also δ will denote constants that depend only on the masses and total energy.

6.3.1. *Bound 1: On the Energy H_1.* Take ρ so large that

$$\beta_1/\rho + g \leq c/\rho \tag{6.14}$$

where c is a prefixed constant. (Any constant larger than β_1 will do, the closer to β_1, the larger we will have to take ρ to be.) Since $H_2 = (pos.) - \beta_2/\rho$ and $H_1 = H - H_2 + g = H - (pos.) + \beta_2/\rho + g$ we have that

$$H_1 \leq H + c/\rho \tag{6.15}$$

Now assume that the total energy H is negative. We obtain

$$H_1 \leq \left(1 - \frac{c}{\rho|H|}\right) H < -c_1|H| \tag{6.16}$$

where for c_1 we take any constant less than $1 - (c/\rho|H|)$. (So c_1 can be made arbitrarily close to 1 by taking ρ large.)

6.3.2. *Bound 2: On r, the bound distance.* From $H_1 = (pos.) - \beta_1/r$ and the previous bound (6.16) we have that $\beta_1/r \geq c_1|H|$. It follows that

$$r \leq \beta_1/c_1|H| := c_2 \tag{6.17}$$

6.3.3. *Bound 3: On the separate angular momenta.* Multiply H_1 through by r and use $H_1 < 0$ (see equation (6.16) to see that

$$\frac{\alpha_1}{2} r|\dot{\xi}_1|^2 \leq \beta_1.$$

Multiply this last inequality through by $2\alpha_1 r$, use inequality 6.17 and take a square root to obtain

$$\alpha_1|\xi_1||\dot{\xi}_1| \leq \beta_1\sqrt{2\alpha_1/(c_1|H|)} \tag{6.18}$$

But $J_1 = \alpha_1\xi_1 \wedge \dot{\xi}_1$ and $|\xi_1 \wedge \dot{\xi}_1| \leq |\xi_1||\dot{\xi}_1|$ so this last equation yields

$$|J_1| \leq \beta_1\sqrt{2\alpha_1/(c_1|H|)} := c_3 \tag{6.19}$$

To bound J_2 use $J = J_1 + J_2 = 0$ to obtain the same bound for J_2:

$$|J_2| \leq c_3 \tag{6.20}$$

6.3.4. *Bound 4: On the far transverse velocity.* Decompose $\dot{\xi}_2$ into radial and transverse velocities:

$$\dot{\xi}_2 = \nu\hat{\xi}_2 + V_2^{\perp}$$

where $\hat{\xi}_2 = \xi_2/\rho$ is the unit vector in the ξ_2 direction and where $V_2^{\perp}$ is orthogonal to ξ_2. Note that

$$\nu = \dot{\rho} = \left\langle \dot{\xi}_2, \hat{\xi}_2 \right\rangle$$

Now $|\xi_2 \wedge \dot{\xi}_2| = \rho|V_2^{\perp}|$, so that inequality (6.20) implies that

$$|V_2^{\perp}| \leq c_4/\rho \tag{6.21}$$

with $c_4 = c_3/\alpha_2$.

We now move to the **integral bounds**.

6.3.5. *Bound 5: An approximating Kepler equation for ρ.* We have that $\dot{\rho} := \left\langle \hat{\xi}_2, \dot{\xi}_2 \right\rangle$. so that

$$\ddot{\rho} = \left\langle \frac{d}{dt}\hat{\xi}_2, \dot{\xi}_2 \right\rangle + \left\langle \hat{\xi}_2, \ddot{\xi}_2 \right\rangle. \tag{6.22}$$

Now

$$\left\langle \frac{d}{dt}\hat{\xi}_2, \dot{\xi}_2 \right\rangle = -\frac{\dot{\rho}^2}{\rho} + \frac{\|\dot{\xi}^2\|}{\rho} = \frac{\|V^{\perp}\|^2}{\rho}$$

so that from inequality (6.21) we have

$$\left| \left\langle \frac{d}{dt}\hat{\xi}_2, \dot{\xi}_2 \right\rangle \right| \leq c_4^2/\rho^3 \tag{6.23}$$

Now use Newton's equations for ξ_2

$$\alpha_2\ddot{\xi}_2 = U_{\xi_2} = -\beta_2\xi_2/\rho^3 + g_{\xi_2}.$$

which yields

$$\ddot{\xi}_2 = -M\xi_2/\rho^3 + \frac{1}{\alpha_2}g_{\xi_2}$$

($\beta_2/\alpha_2 = M = m_1 + m_2 + m_3$). Thus

$$\left\langle \hat{\xi}_2, \ddot{\xi}_2 \right\rangle = -M/\rho^2 + \frac{1}{\alpha_2}\left\langle \hat{\xi}_2, g_{\xi_2} \right\rangle. \tag{6.24}$$

Using the estimate (6.13) on the gradient g_ξ of g, with equations (6.22), (6.23), and (6.24) we get

$$|\ddot{\rho} + M/\rho^2| \leq c_5/\rho^3 \tag{6.25}$$

So that

$$-(M + a)/\rho^2 < \ddot{\rho} < -(M - a)/\rho^2 \tag{6.26}$$

where a is small constant, and where the inequality is valid as long as $c_5/\rho < a$. Note a can be made arbitrarily small by taking ρ large.

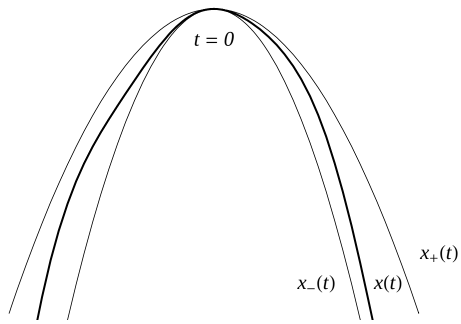

FIGURE 5. Functions in the Sandwich Lemma (6.2)

6.3.6. *Bound 6: Bounding ρ; Fact* $1'$. Here we finally prove the needed Fact $1'$.

Let (A, B) be the largest time interval containing $t = 0$ such that $\rho(t) \geq \rho_0/2$. Here we suppose that $\rho(0) = \rho_0$ is so large that the first five bounds all hold, with $\rho = \rho_0/2$. Then, as long as t lies in (A, B), the previous five bounds all hold. Write $\nu_0 = \dot{\rho}(0)$.

Claim 6.1.

$$|A|, |B| \geq \frac{1}{2}\frac{\rho_0}{|\nu_0|} + O\left(\frac{1}{\nu_0^3 \rho_0}\right) = c_6 \frac{1}{\epsilon} + O(\epsilon) \tag{6.27}$$

It follows from the claim that if $\nu_0 = O(1)$, then $\rho(t) \geq \rho_0/2$ for all t in an interval of the form $[-c_6/\epsilon, c_6/\epsilon]$, with $\epsilon = 1/\rho_0$. In other words, we will have established the needed Fact $1'$.

For the proof we will need a sandwich lemma which the reader can find in [14] where it was called 'The comparison lemma'. The lemma sandwiches $\rho(t)$ between solutions $\rho_-(t) < \rho_+(t)$ to Kepler's problems from the previous bound, i.e. the one-dimensional Kepler equations with Kepler constants $M + a$ and $M - a$. This sandwiching holds provided $c_5/\rho < a$, but the sandwich lemma itself will enforce this inequality over a time domain determined by the solution to Kepler's equation.

Lemma 6.2. *[Sandwich Lemma]. Consider three scalar differential equations $\ddot{x}_- = F_-(x_-)$, $\ddot{x} = F(x, t)$, $\ddot{x}_+ = F_+(x_+)$ with C^1 right hand sides satisfying $F_-(x) < F(x, t) < F_+(x) < 0$ for $x > x_c$, x_c a fixed constant. Suppose that $F_-(x)$ and $F_+(x)$ are monotone increasing for $x > x_c$ Let $x_-(t)$, $x(t)$, $x_+(t)$ be the solutions to their respective differential equation which share initial conditions at $t = 0$: $x_-(0) = x_1(0) = x_+(0) := x_* > x_c, \dot{x}_-(0) = \dot{x}(0) = \dot{x}_+(0)$. Then, for all times t such that $x_-(t) \geq x_c$ we have*

- *(1) $x_-(t) \leq x(t) \leq x_+(t)$ with equality only at $t = 0$, and*
- *(2) $dx_-(t)/dt < dx(t)/dt < dx_+(t)/dt$ for $t > 0$ and*
- *(3) $dx_-(t)/dt > dx(t)/dt > dx_+(t)/dt$ for $t < 0$*

See Figure 5.

The Sandwich Lemma is proved in [14] where it is called a Comparison Lemma. Applying the lemma with $F_\pm(\rho) = -(M \mp a)/\rho^2$ yields $\rho_-(t) < \rho(t) < \rho_+(t)$. We will only need the lower bound. Referring to our interval (A, B) for ρ, let $(A_-(\nu_0), B_-(\nu_0))$ be the analogous interval for the lower bound solution ρ_-. By the comparison lemma we have that $(A_-(\nu_0), B_-(\nu_0)) \subset (A, B)$, so that it suffices to show the validity of the estimate (6.27) with $A_-(\nu_0), B_-(\nu_0)$ in place of A, B.

It is clear that $B_-(\nu_0)$ is monotone increasing function of ν_0: the faster you head for infinity initially, the longer it takes you to return. This assertion remains true for negative ν_0. Hence, to make B_- as small as possible, we should have $\nu_0 < 0$ and large. With this in mind, we now suppose that $\nu_0 < 0$, and take it to be large if need be. Then $\ddot{\rho} < 0$ over our interval, so that for any time t with $O < t < B_-$ we have $-\dot{\rho}(t) > -\nu_0$. From $dt = \frac{dt}{d\rho} d\rho$ it follows that $B_- = \int_0^{B_-} dt = \int \frac{1}{|\dot{\rho}_-|}|d\rho| = \int \frac{d\rho}{-\dot{\rho}_-}$. Write $H_K = \frac{1}{2}\dot{\rho}_-^2 - \frac{M+a}{\rho_-}$ for the Kepler energy associated to the lower Kepler bound. Then we have

$$\begin{aligned}
-\dot{\rho}_-(t) &= \sqrt{2\left(H_K + \frac{M+a}{\rho_-(t)}\right)} \\
&= \sqrt{\nu_0^2 - 2\frac{M+a}{\rho_0} + 2\frac{M+a}{\rho_-(t)}} \\
&= |\nu_0|\sqrt{1 + \frac{2(M+a)}{\nu_0^2}\left(\frac{1}{\rho_-} - \frac{1}{\rho_0}\right)} \\
&= |\nu_0|\left(1 + O\left(\frac{1}{\nu_0^2\rho_0^2}\right)\right)
\end{aligned}$$

Plugging this estimate for $\dot{\rho}$ into the integrand of :

$$B_- = -\int_{\rho_0}^{\rho_0/2} \frac{d\rho}{-\dot{\rho}}$$

we obtain

$$B_- = \frac{1}{2}\frac{\rho_0}{|\nu_0|} - O\left(\frac{1}{\nu_0^3\rho_0}\right)$$

A time-symmetric argument yields the same bound for $|A_-|$.

6.3.7. *Bound 7: On the variation of H_1.* We claim that for $|t| < c_6/\epsilon$ we have that

$$|H_1(t) - H_1(0)| \leq c_7/\rho_0^2 \tag{6.28}$$

We have $H_1 = H - H_2 - g$. The total energy H is constant. The "perturbation" g satisfies $|g| \leq c/\rho_0^3$ over the interval in question so that its variation is at most $2c/\rho_0^3$. It follows that bound (6.28) is equivalent to

$$||H_2(t) - H_2(0)|| \leq c_7/\rho_0^2 \tag{6.29}$$

Now

$$H_2 = \frac{\alpha_2}{2}\left(\dot{\rho}^2 + |V_2^{\perp}|^2\right) - \frac{\beta_2}{\rho}$$

and we have seen that

$$|V_2^{\perp}|^2 \leq c_4/\rho_0^2$$

so that, the variation of $|V_2^{\perp}|^2$ is also $O(1/\rho_0^2)$. Thus to establish (6.29) we must establish that the variation of the "radial part"

$$H_2^{\rho} = \frac{\alpha_2}{2}\dot{\rho}^2 - \frac{\beta_2}{\rho}$$

of H_2 is of order c/ρ_0^2. We have:

$$\frac{d}{dt}H_2^{\rho} = \dot{\rho}\left(\alpha_2\ddot{\rho} + \frac{\beta_2}{\rho^2}\right)$$

so that, from eq (6.25) we have

$$\left| \frac{d}{dt} H_2^\rho \right| \le |\dot\rho|\, c_5/\rho^3$$

Integrating:

$$|H_2^\rho(t) - H_2^\rho(0)| \le c_5 \int_{\rho_0}^{\rho(t)} \frac{|d\rho|}{\rho^3} = c_7/\rho_0^2,$$

where $c_7 = c_4/2$, and where in the last equality we used that $\frac{1}{2}\rho(t) \le \rho_0 \le 2\rho(t)$ for $|t| < c_6/\epsilon$. This last inequality is the desired result.

6.4. **Appendix 4: The Levi-Civita transform.** In this Appendix we show how the Levi-Civita transform (4.3) transforms a perturbed Kepler equation into a perturbed harmonic oscillator. This fact is well-known and can be found in [17], [8] , or [1]. Even so, we present the derivation for completeness, and so as to render the Levi-Civita transformation and its outcome in the coordinates of this paper.

Consider the perturbed planar Kepler problem:

$$\ddot\xi = -\gamma\xi/r^3 + f \tag{6.30}$$

where γ is a constant , $r = |\xi|$ and f is the perturbation, considered to be small, and allowed to be a function of $\xi, \dot\xi$ and t. The Levi-Civita transform (see (4.3) is the simultaneous change of dependent (ξ)and independent (t) variables defined by $z^2 = \xi$ and $ds = |dt/r|$. We show that that the Levi-Civita transform converts our perturbed Kepler problem into the perturbed oscillator:

$$z'' = \frac{1}{2} H z - \frac{1}{2}\bar z^2 z f.$$

Here the prime denotes d/ds and

$$H_1 = \frac{1}{2}|\dot\xi|^2 - \frac{\gamma}{|\xi|}.$$

is the unperturbed Kepler energy.

Write $'$ for d/ds and $\dot{}$ for d/dt. Since $d/dt = r^{-1} d/ds$ we have $\dot\xi = r^{-1}\xi'$ and

$$\begin{aligned}
\ddot\xi &= \frac{d}{dt}\left(\frac{1}{r}\xi'\right) && (6.31)\\
&= \frac{1}{r}\frac{d}{ds}\left(\frac{1}{r}\xi'\right) && (6.32)\\
&= \frac{1}{r^2}\xi'' - \frac{r'}{r^3}\xi' && (6.33)\\
& && (6.34)
\end{aligned}$$

It follows that

$$r^3\ddot\xi = r\xi'' - r'\xi'$$

Thus, upon multiplying both sides of (6.30) by r^3 and rearranging we get

$$r\xi'' - r'\xi' + \gamma\xi = r^3 f \tag{6.35}$$

Now, from $\xi = z^2$ we get

$$\xi' = 2zz'$$

while

$$\xi'' = 2zz'' + 2z'^2$$

Also, from

$$r = |z|^2$$

we get

$$r' = 2\langle z, z'\rangle$$

Thus

$$r\xi'' - r'\xi' = 2|z|^2 zz'' + 2|z|^2 z'^2 - 4\langle z, z'\rangle zz'$$

and (6.35) becomes

$$2|z|^2 zz'' + 2|z|^2 z'^2 - 4\langle z, z'\rangle zz' + \gamma z^2 = -|z|^6 f$$

We now divide through by $2|z|^2 z$ to get

$$z'' + \frac{z'^2}{z} - 2\frac{\langle z, z'\rangle}{|z|^2} z' + \frac{\gamma}{2|z|^2} z = -\frac{|z|^4}{2z} f$$

or:

$$z'' + \left\{\frac{z'^2}{w^2} - 2\frac{\langle z, z'\rangle}{|z|^2}\frac{z'}{z} + \frac{\gamma}{2|z|^2}\right\} z = -\frac{|z|^4}{2z} f$$

Now, we claim that

$$-\frac{1}{2}H_1 = \left\{\frac{z'^2}{z^2} - 2\frac{\langle z, z'\rangle}{|z|^2}\frac{z'}{z} + \frac{\gamma}{2|z|^2}\right\}$$

a fact which, once established will complete the computation, since $|z|^2 = z\bar{z}$, so that that

$$\frac{|z|^4}{2z} f = \frac{1}{2}\bar{z}^2 z f.$$

Now $\gamma/2|\xi| = \gamma/(2|z|^2)$ is half the potential energy term of H_1. Thus, we are done with our computation, once we have established the rather surprising identity

$$-\frac{1}{2}K.E. = \frac{z'^2}{z^2} - 2\frac{\langle z, z'\rangle}{|z|^2}\frac{z'}{z}$$

where $K.E. = \frac{1}{2}|\dot{\xi}|^2 = 2|z\dot{z}|^2$ is the kinetic energy term of H_1. This kinetic energy identity is achieved after a fair amount of algebra, best done using polar coordinates $z = \rho e^{i\theta}$.

6.5. **Appendix 5: An oscillator Gronwall inequality; validity of the Kepler approximation.** We consider a perturbed oscillator $z(\tau)$:

$$z'' = -\omega_0^2 z + \epsilon^2 f(z, z', \tau; \epsilon) \tag{6.36}$$

with $\omega_0 > 0$ constant and $|f| \leq C_f$ for $|\tau| < C_f/\epsilon$. We will compare the solutions $z(\tau)$ of this equation to the solutions $z_0(\tau)$ of the unperturbed oscillator

$$z_0'' = -\omega_0^2 z_0 \tag{6.37}$$

Lemma 6.3. *There exist constants C_1, C_2 such that solutions to (6.36) and (6.37) having the same initial conditions at $\tau = 0$ lie within $C_2\epsilon$ of each other for times $|\tau| < C_1/\epsilon$; that is to say, the bound (4.6) holds: $|z(\tau) - z_0(\tau)| < C_2\epsilon$. (The constants C_1, C_2 depend only on C_f and on the value of the unperturbed energy $\frac{1}{2}(|z'|^2 + \omega_0^2|z|^2)$.)*

Proof. We may suppose that $z_0(\tau)$, the solution to (6.37), is non-zero, the contrary case being rather immediately established.

In a neighborhood of the curve $z_0(t)$ we can find Hopf-type coordinates $w_1, w_2, w_3, w_0 = \theta$ such that $z_0(t)$ is the curve $w_1 = w_2 = w_3 = 0$ and such that the equations for the unperturbed oscillator (6.37) become

$$w_1' = 0$$

$$w_2' = 0$$

$$w_3' = 0$$

$$\theta' = \omega_0$$

To see these coordinates explicitly in case $\omega_0 = 1$ observe that the Hamiltonian for the unperturbed oscillator is $H_0 = \frac{1}{2}(|z|^2 + |z'|^2)$ so that its level sets are three-spheres within the $\mathbb{C}^2$ coordinatized by (z, z'). The oscillator flow is precisely the Hopf flow, but relative to the complex structure with coordinates $z_1 = x + ix', z_2 = y + iy'$, where $z_0 = x + iy$. For w_1, w_2, w_3, we can use the standard Hopf projection coordinates,$w_1 = \frac{1}{2}(|z_1|^2 - |z_2|^2)$, $w_i + iw_2 = z_1\bar{z}_2$. A local trivialization of the Hopf fibration will yield the fiber coordinate θ.

These coordinates are valid in a (large) neighborhood of an unperturbed orbit $(z_0(\tau), z_0'(\tau))$. In these coordinates the perturbed equations become:

$$w_1' = \epsilon^2 f_1$$

$$w_2' = \epsilon^2 f_2$$

$$w_3' = \epsilon^2 f_3$$

$$\theta' = \omega_0 + \epsilon^2 f_4$$

where the coordinate vector field (f_1, f_2, f_3, f_4) is the vector field of f written out in the w-coordinates. Since the Jacobian of the coordinate transformation to the w's is bounded in a neighborhood of the unperturbed orbit, there will be another constant $\widetilde{C}_f$ such that the bound $|f| \leq \widetilde{C}_f$ holds for the new coordinate vector field $f = (f_1, f_2, f_3, f_4)$. Set $w_4 = \theta$ and $w = (w_1, w_2, w_3, w_4)$. Write w^0 for the vector $w(\tau)$ associated to $(z_0(\tau), z_0'(\tau))$, and $w(\tau)$ for $(z(\tau), z'(\tau))$. Then,

$$w' - w^{0\prime} = \epsilon^2 f.$$

It follows that

$$|w' - w^{0\prime}| \leq \epsilon^2 \tilde{C}_f$$

Integration, plus $w(0) - w^0(0) = 0$ yields

$$|w(\tau) - w^0(\tau)| \leq \epsilon^2 \tilde{C}_f |\tau|$$

so that

$$|w(\tau) - w^0(\tau)| \leq \epsilon \tilde{c}_{10}$$

provided that $|\tau| \leq \tilde{C}_\tau/\epsilon$, and where the constant $\tilde{c}_{10} = \widetilde{C}_f \widetilde{C}_\tau$. Finally, in a neighborhood of the orbit, the coordinate norms associated with z and w are Lipschitz equivalent, so we get the desired bound (4.6) □

6.5.1. *Proof of the validity of Kepler approximation.*

Proof. We continue the notation from immediately above.

Let N be given, as in the theorem. Let $P = \pi/\omega_0$ be the oscillator's half period. Here $\omega_0 = \sqrt{|H_1(0)|/2}$ is the instantaneous frequency. Write $P_K = \int_0^P |z_0(\tau)|^2 d\tau$ for the instantaneous Kepler period. (It is not a half period, due to the double cover feature of the Levi-Civita transformation.) According to the Gronwall lemma above that there are constants C_f, C_1, C_2 of order 1 in ϵ such that if both conditions

- (i) $|f(t)| < C_f$ and
- (ii) $|\tau| < C_1/\epsilon$

hold, then

$$|z(\tau) - z_0(\tau)| < C_2\epsilon \tag{6.38}$$

holds. (The times t and τ are related by the Levi-Civita transform.) From the work of Appendix 3, there is a c_6 such that if $|t| < c_6/\epsilon$ then $|f(t)| \le c_4$. Take C_* to be the minimum of the constant C_1, C_2, c_6, c_4, C_f. □

Claim 6.4. Set $\Delta = NP$. If

$$\epsilon < \min\{C_*/\Delta, C_*/(NP_K + C_2), 1/\Delta\} \tag{6.39}$$

then

$$|z(\tau) - z_0(\tau)| < C_2\epsilon \tag{6.40}$$

holds for all $|\tau| < NP$.

This claim will establishes the validity of the Kepler approximation (4.6) as we need it in Section 4.

6.5.2. *Proof of Claim.*

Proof. Let τ_* be the supremum of the numbers such that (6.40) hold for all $|\tau| < \tau_*$. We are to show that $\tau_* \ge \Delta = NP$. If $\tau_* < \infty$, then by continuity we must have equality $|z(\tau) - z_0(\tau)| = C_2\epsilon$ at one of the two numbers $\tau = \pm\tau_*$.

By definition, if $|\tau| < \tau_*$ then (6.40) holds, and so:

$$\begin{aligned}
|t| &= \left|\int_0^\tau r(\tau)d\tau\right| \\
&= \left|\int_0^\tau |z(\tau)|^2 d\tau\right| \\
&< \left|\int_0^\tau (|z_0(\tau)|^2 + C_2\epsilon)d\tau\right| \\
&\le \left|\int_0^\tau (|z_0(\tau)|^2 d\tau| + \epsilon C_2 \int_0^\tau d\tau\right| \\
&\le nP_K + \epsilon C_2|\tau|
\end{aligned}$$

where the integer n is the first positive integer such that $|\tau| \le nP$. In other words, if τ/P is not integer, then n is 1 plus the integer part of τ/P. In the last line we used the periodicity of $z_0(\tau)$ and the relation between P and P_K.

Thus if $|\tau| < \Delta$ and $n \le N$ we have $|t| < NP_K + \epsilon C_2\Delta$. And since $\epsilon < 1/\Delta$ we have $\epsilon C_2\Delta < C_2$, so that $NP_K + \epsilon C_2\Delta < NP_K + C_2 < C_*/\epsilon < c_6/\epsilon$ from our choice of ϵ. Because $|t| < c_6/\epsilon$ we have that $|f(t)| < C_f$.

We can finish off by contradiction. If $\tau_* < \Delta$ we have just seen that the condition $|f(t)| < C_f$ holds all the way up to the times t_* corresponding under Levi-Civita to $\pm\tau_*$. But from $\epsilon < C_*/NP < C_1/\Delta$ we have $\Delta < C_1/\epsilon$ so that $\tau_* < C_1/\epsilon$ so both conditions (i) and (ii) above hold, and hence the inequality (6.40) is strict even up to the Levi-Civita times $\pm\tau_*$. This contradicts the definition of τ_*. □

6.6. **Appendix 6: A perturbed harmonic oscillator with stutters.** This paper began with the question: Do there exist large regions of phase space within which any solution is stutter-free? Focussing on the region $\rho >> r$ and using the Levi-Civita transform (Appendix 4) converts this question to the following question. For a perturbed planar harmonic oscillator is there a (small) neighborhood of the origin with the property that any solution starting in that neighborhood and not colliding with the origin must wind around the origin? If the answer had been 'True' then our theorem would be false: there would have been no stutters of the type described in the theorem. That the answer is 'False' is shown by the following example.

Consider the perturbed harmonic oscillator

$$\ddot{x} = -x$$

$$\ddot{y} = -y + \epsilon f(t)|z|x$$

where $f(t) = 1$ for $0 < t < \pi/2$, and $f = 0$ for $\pi/2 < t < \pi$, and extend f to be 2π periodic. The significance of $\pi/2$ is that it is the oscillator's quarter period. The function f can be smoothed off with the same results.

The perturbing vector field $\epsilon f(t)|z|(0, x)$ is a shear field parallel to the y-axis and turns on or off with a period 1/4 that of the oscillator's. This perturbation 'torques' the unperturbed solutions counterclockwise, destroying the winding of solutions which start on the positive y-axis perpendicular to that axis, i.e solutions with initial conditions $z(0) = (0, y_0), \dot{z}(0) = (\delta, 0), \delta > 0$. The smaller the initial velocity δ, the closer the solution start to the origin, i.e. the smaller y_0 must be to destroy the winding. A solution which is precariously close to the origin at $t = 0$ will acquire the "wrong" angular momentum during its time away from the origin and its winding is destroyed. Instead, it will "snake" its way up the y-axis, never winding around the origin.

Here is a detailed analytic realization of our example:

$$\begin{aligned} \ddot{x} &= -x \\ \ddot{y} &= -y + \epsilon f(t) x^a \end{aligned}$$

$a \geq 2$ any integer. The function $f(t)$ is as above f: or, more generally, any nonnegative integrable function, f whose support lies in $[0, \pi/2]$ and which is positive on a set of positive measure. For example, we could make f a smooth positive version of a step function by making it 2π periodic, insisting that $f \equiv 1$ for $\mu_1 \leq t \leq \pi/2 - \mu_2$ that $f \equiv 0$ for t outside of $[0.\pi/2]$, with f smooth and monotone decreasing. Here $\mu_1, \mu_2 < \pi/2$ are positive numbers. Take the same class of initial conditions as above, with $y_0, \delta > 0$.

Solve the x ODE to get: $x(t) = \delta \sin(t)$. Plug this expression for x into the y equation to get

$$\ddot{y} = -y + \epsilon\delta f(t) \sin(t)^a \qquad (*)$$

which can be solved by the method of variation of parameters.

Acknowledgements. We would like to thank Carles Simo for his enthusiasm and interest in our work and we wish him many more years of enormous energy. We would also like to thank the Bureau des Longitudes in Paris, and in particular Jacques Fejoz, but also Alain Albouy, Alain Chenciner, and Jacques Laskar for numerous conversations and an invitation to speak on this topic, and Rick Moeckel of the U. of Minnesota for helpful comments.

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Received October 2006; revised November 2007.

E-mail address: skaplan@unca.edu
E-mail address: levi@math.psu.edu
E-mail address: rmont@ucsc.edu