A Boat Paradox

MATHEMATICAL CURIOSITIES
By Mark Levi

Here is a twist on the standard high school boat problem (a topic appropriate for the approaching summer season). In the usual version, a boater walks from stern to bow, causing the boat to slide backward; the question is to determine the sliding distance, ignoring the drag. (The answer is $\frac{m}{m+M}L$, where L is the length of the boat and where m, M are the masses of the person and of the boat.) In the present "twisted" version the setting is identical, except that the drag is included and *assumed to be linear in the velocity.* The drag coefficient k, as well as the masses m, M, are given. The boat and the person are initially at rest, and the question remains the same: what is the boat's eventual displacement from its initial position?

Remarkably, the answer is zero. After the person stops moving, the boat glides, approaching its *initial position* as $t \to \infty$. The values of m, M and k are irrelevant, as long as $k \neq 0$.

Here is the proof. Let x denote the position of the center of mass of the system (boat + person) on the horizontal axis in the figure, and let b denote the position of the boat, e.g., of its bow. The drag force on the boat is then $-k\dot{b}$, and by Newton's second law

$$(m+M)\ddot{x} = -k\dot{b}.$$

In other words, we have

$$\dot{b} = c\ddot{x}, \qquad (1)$$

where $c = (m+M)/k$, a value that is irrelevant as long as it is finite, i.e. as long as $k \neq 0$. Integration by t gives

$$b\Big|_0^\infty = c\dot{x}\Big|_0^\infty .$$

But $\dot{x}(0) = \dot{x}(\infty) = 0$ because everything is at rest before and after, and we conclude that $b(\infty) - b(0) = 0$, proving that the boat ends in its initial position, as claimed. And the values of m, M and k are immaterial, *as long as* $k \neq 0$.

Figure 1 shows the stages between $t=0$ and $t=\infty$. The person starts walking forward, causing the boat to start sliding backward. But backward sliding implies the forward drag, and thus the forward acceleration of the center of mass (this is a bit like starting to walk on ice: the feet slide backward, but the center of mass accelerates forward). To summarize, the boat moves back, but the center of mass moves forward. Once the person stops, the center of mass—together with the hull—continues to glide forward. In the limit of $t \to \infty$, this gliding exactly restores the earlier backward displacement of the boat.

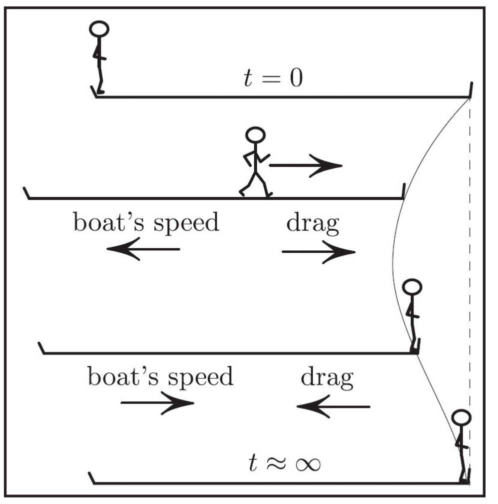

Figure 1: *The inevitable return to initial position.*

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