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A Lagrangian Proof of the Invariant Curve Theorem for Twist Mappings

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1. Background on generating functions

This proof is contained in unpublished lecture notes of a course held by J. Moser at the ETH-Zürich in June 1986.

In contrast to the original "Hamiltonian" approach, where the invariant curve is sought in the phase space, the present Lagrangian approach is based on reducing the problem of finding an invariant curve to solving a nonlinear second difference equation in the configuration space. This approach is modeled after the proof of an analogue theorem for elliptic partial differential equations [**9**] where it was needed since the transformation technique fails for partial differential equations. In our situation the invariance of the desired curve under a mapping leads to a nonlinear second order difference equation which will be derived in the next section and will be solved in the subsequent sections.

Let us consider an area–preserving twist mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the covering plane of the cylinder $\mathbb{R}(\text{mod } 1) \times \mathbb{R}$. Assume that $\varphi : (x_1, y_1) \to (x_2, y_2)$ has a generating function h:

$$\begin{aligned} h_1(x_1, x_2) &= -y_1, \\ h_2(x_1, x_2) &= \ \ y_2. \end{aligned} \tag{1}$$

Here h_1, h_2 are the derivatives with respect to the first or second argument of h. Similarly we will denote the second derivatives by h_{11}, h_{12}, h_{22}. The following theorem establishes a sufficient condition under which a mapping is given by (1).

THEOREM 1. *Any smooth twist cylinder map φ satisfying the monotone twist condition $\partial x_2/\partial y_1 > 0$ possesses a generating function h such that the map is given by (1) implicitly. Moreover, the map is exact: $\int_{\varphi\gamma} y dx = \int_\gamma y dx$, where γ is an arbitrary smooth noncontractible circle on the cylinder, iff $h(x_1 + 1, x_2 + 1) = h(x_1, x_2)$ and $h_{12} < 0$.*

2. Reduction to a difference equation

The existence of an invariant curve is equivalent to a second order differential equation; this is the main point of this section.

Consider a closed curve wrapping around the cylinder, with parametric equations $x = u(\theta)$, $y = v(\theta)$ in the covering (x, y)–plane of the cylinder, with $u(\theta) - \theta$ and $v(\theta)$ periodic of period 1, Figure 1.

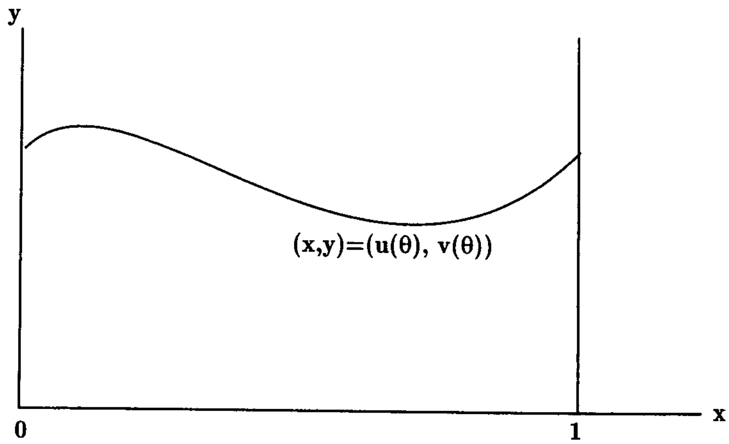

FIGURE 1. The invariant curve

We will find an invariant curve which satisfies the invariance condition

$$\varphi(w(\theta)) = w(\theta + \omega), \tag{2}$$

with a prescribed rotation number ω. Moreover, u is assumed to be strictly monotone in θ. According to (2) the restriction of φ to the curve is conjugate to a rotation by ω.

The following is a Lagrangian formulation of the invariance condition (2), as it occurs in Mather's theory.

THEOREM 2. *The curve $w(\theta) = (u(\theta), v(\theta))$ satisfies the invariance condition (2) with φ given by (1) iff the horizontal function $u(\theta)$ satisfies the second order difference equation*

$$E(u(\theta)) \equiv h_1(u(\theta), u(\theta + \omega)) + h_2(u(\theta - \omega), u(\theta)) = 0. \tag{3}$$

To prove the theorem we simply shift θ by $-\omega$ in the second equation in (1) and add the two equations in (1).

The Hamiltonian problem of finding $w(\theta)$ has thus been reduced to the Lagrangian equation for a single function $u(\theta)$. If $u(\theta)$ has been so determined one finds $v = v(\theta)$ from the formula $v = -h_1(u, u^+)$; here and in the following we use the abbreviation $u^+ = u(\theta + \omega)$, $u^- = u(\theta - \omega)$.

REMARK 3. The equation (3) is the Euler–Lagrange variational equation for the variational problem $\delta \int_0^1 h(u, u^+) d\theta = 0$ (Percival's variational principle).

REMARK 4. The mean value of $u_\theta E(u)$ is zero:

$$\int_0^1 u_\theta E(u) d\theta = 0 \tag{4}$$

as follows from the invariance of the Lagrangian $\int_0^1 h(u,u^+)d\theta$ under θ-translations $u(\theta) \to u(\theta + c)$ or from the identity

$$u_\theta E(u) = \frac{\partial}{\partial\theta} h(u,u^+) - \nabla(u_\theta h_2(u^-,u))$$

where $\nabla f = f(\theta+\omega) - f(\theta)$. Integration gives the claim.

EXAMPLE 5. Consider the standard map

$$\begin{aligned} x_2 &= x_1 + y_1 + S'(x_1) \\ y_2 &= \qquad\; y_1 + S'(x_1) \end{aligned}$$

the generating function is

$$h(x_1,x_2) = \frac{1}{2}(x_1 - x_2)^2 + S(x_1)$$

with a periodic function S of period 1 the equation (3) takes the form

$$u(\theta+\omega) - 2u(\theta) + u(\theta-\omega) = S'(u(\theta)).$$

3. Main theorem

The invariant curve theorem stated here is in effect an implicit function theorem stating that in the vicinity of a near-solution: $E(u_0) \approx 0$ there exists an exact solution: $E(u) = 0$ provided, in particular, that ω is Diophantine.

Notations. Let W_r be the set of 1-periodic real analytic functions of θ bounded on the strip $|\mathrm{Im}\theta| \le r$. Introduce the maximum norm

$$|f|_r \equiv \sup_{|Im\theta|\le r} |f(\theta)|.$$

Assumptions. We assume that h is analytic for $(x_1,x_2) \in \mathcal{D} \subset \mathbf{C}^2$ real for (x_1,x_2) and invariant under translations mentioned in Theorem 1. For $R > 0$, let $\mathcal{D}_R \subset \mathcal{D}$ denote the largest set whose R-neighborhood lies in $\mathcal{D}$, Figure 2.

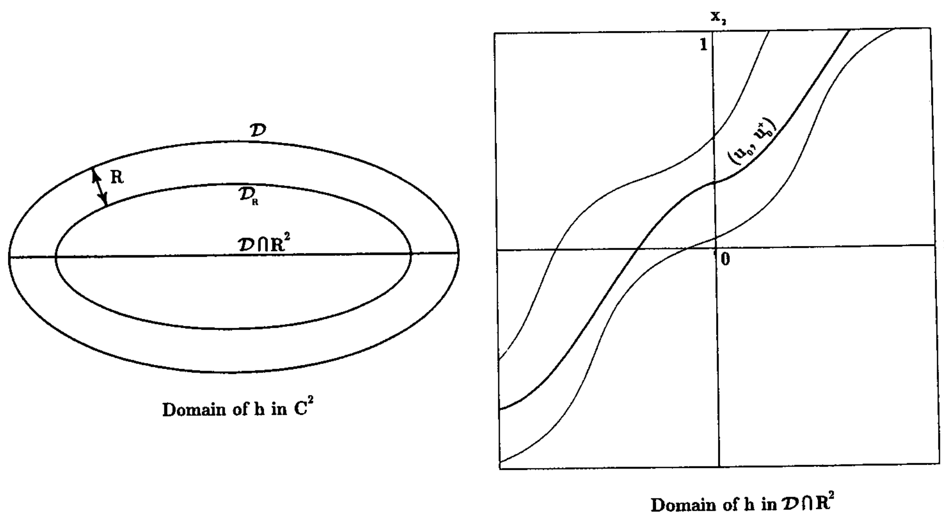

FIGURE 2. Schematic image of the domain of h

Thus the points in $\mathcal{D}_R$ are "safely" inside $\mathcal{D}(h)$. We assume that

$$\min_{\mathcal{D}}|h_{12}| > \kappa, \quad \kappa > 0, \tag{5}$$

and that there exists $M > 0$ such that

$$|h|_{C^3(\mathcal{D})} < M. \tag{6}$$

Let $u_0(\theta) - \theta \in W_r$ for some $0 < r < 1$. Assume, moreover, that

$$(u_0, u_0^+) \in \mathcal{D}_R \text{ for } |\mathrm{Im}\theta| < r, \tag{7}$$

so that (u_0, u_0^+) is "safely" inside the domain of h, Figure 2, and that for some $N_0 > 0$ (possibly large)

$$|(u_0)_\theta|_r < N_0, \ |(u_0)_\theta^{-1}|_r < N_0. \tag{8}$$

THEOREM 6 (Main theorem). *Assume that ω is Diophantine: there exist $K > 0$ and $\sigma > 0$ such that*

$$|\omega - \frac{p}{q}| \geq \frac{K}{q^{2+\sigma}} \quad \forall \text{ integer } p, q \neq 0. \tag{9}$$

Assume that h and u_0 satisfy the above conditions. There exists $\delta = \delta(r, h, M, N_0, K, \sigma, \kappa)$ such that if $|E(u_0)|_r < \delta$, then there exists a unique solution u near u_0 of $E(u) = 0$ with $u(\theta) - \theta \in W_{r/2}$ and mean value of $u(\theta) - \theta$ equal to zero.

Application to the near-integrable twist map. Consider a small perturbation of the linear twist map:

$$\begin{aligned} x_2 &= x_1 + y_1 + \varepsilon f(x_1, y_1, \varepsilon) \\ y_2 &= y_1 + \varepsilon g(x_1, y_1, \varepsilon), \end{aligned} \tag{10}$$

assumed to be area-preserving and exact. The map is given by a generating function $h(x_1, x_2) = \frac{1}{2}(x_2 - x_1)^2 + \varepsilon H(x_1, x_2, \varepsilon)$ which we assume to be defined on $\mathcal{D} = \{(x_1, x_2) : \ a < \mathrm{Re}(x_2 - x_1) < b, \ |\mathrm{Im}\ x_1| < 1, \ |\mathrm{Im}\ x_2| < 1\}$. To apply the main theorem, choose a Diophantine $a < \omega < b$ and take $u_0(\theta) = \theta$. It is a simple exercise to verify the conditions (5)-(8), along with $|E(u_0)|_r = O(\varepsilon)$. Theorem 6 yields an invariant curve in the annulus $a < y < b$ provided ε is small enough.

REMARK 7. It is important to observe that the smallness assumption $|E(u_0)|_r < \delta$ in the main theorem is more general than the standard near-integrability assumption requiring the map to be of the form (10).

4. The homological equation

We will solve the problem $E(u) = 0$ by a modification of Newton's iteration, starting with u_0. Seeking an improvement $\tilde{u} = u + v$ of u, we write

$$E(u + v) = E(u) + E'(u)v + Q,$$

where Q is the remainder and where

$$E'(u)v = (h_{11} + h_{22}^-)v + h_{12}v^+ + h_{12}^-v^-. \tag{11}$$

The first idea would be to kill linear terms:

$$E'(u)v = -E(u), \tag{12}$$

in order to make $E(u+v)$ quadratically small compared with $E(u)$. This is the basis of the standard Newton iteration. The equation (12) is, however, difficult to solve. Fortunately, it is not necessary to kill linear terms: making them quadratically small would suffice. To determine precisely how to modify the equation (12), we multiply both sides by u_θ and subtract the quadratically small term $v\frac{d}{d\theta}E(u) = vE'(u)u_\theta$ from the left–hand side. The result is the homological equation, no longer equivalent to (12):

$$u_\theta E'(u)v - vE'(u)u_\theta = -u_\theta E(u). \tag{13}$$

When v is determined from this equation, $E(u+v)$ turns out to be still quadratically small compared to $E(u)$ as we shall see later, but first we show how to solve (13). Note that

$$u_\theta E'(u)v - vE'(u)u_\theta = h_{12}(u_\theta v^+ - u_\theta^+ v) + h_{12}^-(u_\theta v^- - u_\theta^- v);$$

introducing now the new unknown $w = \frac{v}{u_\theta}$ we rewrite the above as

$$h_{12}u_\theta u_\theta^+(w^+ - w) - h_{12}^- u_\theta^- u_\theta(w - w^-) = \nabla^*(h_{12}u_\theta u_\theta^+ \nabla w)$$

where

$$\nabla f(\theta) \equiv f(\theta+\omega) - f(\theta), \quad \nabla^* f(\theta) \equiv f(\theta) - f(\theta - \omega).$$

To summarize, we have transformed (13) into

$$\nabla^*(h_{12}u_\theta u_\theta^+ \nabla w) = -u_\theta E(u). \tag{14}$$

The Newton step. is thus defined as follows. Given u, find w from the homological equation (14) and set the updated

$$\tilde{u} = u + v, \;\; v = u_\theta w. \tag{15}$$

In the sections below we will show that with appropriate choices of spaces and constants the repeated iterations of this step converge to the solution of our problem $E(u) = 0$.

An outline of the proof of the main theorem. We will show that one step (15) produces a new error that is quadratic in the preceding one, as one would expect from a Newton-type step: $|E(\tilde{u})|_\rho \sim |E(u)|_r^2/(r-\rho)^{4\tau}$. This is done in section 6. The improvement, however, is counteracted by the small denominator, due to small divisors. In section 7 we show that the step can be iterated infinitely many times and that convergence wins if appropriate choices are made. In the next section 5 we study the homological equation.

A heuristic motivation. The above trick of writing the correction term in the form $u_\theta w$ is essential for this approach. It can be motivated by noticing that, in the case that u is already a solution of $E(u) = 0$, the operator $E'(u)$ has as zero solutions the multiples of u_θ. Thus by setting $v = u_\theta w$ the corresponding operator has the constants as zero solutions ("variation of constants"). Moreover, since the operator E' is symmetric also $u_\theta E'(u)u_\theta$ is symmetric, so that also its adjoint has the constants in its zero space, as is borne out by the formula $u_\theta E'(u)u_\theta w = (\nabla^* a \nabla)w$ (with $a = h_{12}u_\theta u_\theta^+$). Therefore the range of this operator contains only functions of mean value zero and Equation (4) appears as a necessary compatibility condition for the solvability of equation (14).

5. Solving the homological equation

The following lemma gives an estimate on the solution of (14). We treat u as given and w as the unknown.

LEMMA 8. *Assume that $u(\theta)$ satisfies*

$$(u, u^+) \in \mathcal{D}_R \quad for \quad |Im\,\theta| < r, \tag{16}$$

and

$$|u_\theta|_r < N, \; |u_\theta^{-1}|_r < N \quad for \quad |Im\,\theta| < r. \tag{17}$$

Then (14) has a unique solution $w \in W_\rho$, with $w_0 \equiv \int_0^1 w d\theta = 0$ for any $0 < \rho < r$, and the correction $v = u_\theta w$ satisfies the estimates

$$|v|_\rho \leq \frac{c}{(r-\rho)^{2\tau}} |E(u)|_r, \quad |v_\theta|_\rho \leq \frac{c}{(r-\rho)^{2\tau+1}} |E(u)|_r, \tag{18}$$

where $c = c(M, N, K, \sigma)$ and $\tau = 2 + \sigma$.

We will prove this lemma by a two-fold application of the simpler lemma:

LEMMA 9 (Decrease of domain of analyticity). *Let ω satisfy the Diophantine condition (9), and let $g \in W_r$ with mean value $[g] = 0$. The equation*

$$\nabla \psi = g, \tag{19}$$

has a unique solution $\psi \in W_{r'}$ with $[\psi] = 0$ for any $0 < r' < r$. Moreover,

$$|\psi|_{r'} < c(K, \sigma) \frac{|g|_r}{(r - r')^\tau}. \tag{20}$$

PROOF OF LEMMA 9. The Fourier coefficients of g and ψ are related via

$$\psi_n = \frac{g_n}{e^{2\pi i n \omega} - 1} \neq 0, \; \psi_0 = 0. \tag{21}$$

The Diophantine condition (9) gives the lower bound on the denominators:

$$|e^{2\pi i n \omega} - 1| > \frac{c(K)}{|n|^{1+\sigma}}, \tag{22}$$

while $g \in W_r$ implies

$$|g_n| \leq |g|_r e^{-2\pi |n| r}. \tag{23}$$

Using (23) and (22) in (21) we obtain, for $0 < r' < s < r$:

$$|\psi_n| \leq |g|_r c^{-1}(K) e^{-2\pi r |n|} |n|^{1+\sigma} = |g|_r c^{-1}(K) e^{-2\pi s |n|} e^{-2\pi (r-s)|n|} |n|^{1+\sigma}. \tag{24}$$

We estimate the right hand side by using the estimate $xe^{-x} \leq e^{-1}$ for positive x, which for $x = \frac{a|n|}{b}$ gives the inequality $e^{-a|n|}|n|^b \leq e^{-b}(b/a)^b$ for all n; with $a = 2\pi(r - s)$ and $b = 1 + \sigma$ we obtain the result

$$|\psi_n| \leq c_1(K, \sigma) |g|_r \frac{1}{(r-s)^{1+\sigma}} e^{-|n| 2\pi s}.$$

From this estimate on ψ_n we obtain

$$|\psi|_{r'} \leq \sum |\psi_n| e^{|n| 2\pi r'} \leq \frac{2c_1 |g|_r}{(r-s)^{1+\sigma}} \left(1 - e^{-2\pi(s-r')}\right)^{-1} \leq \frac{2c_1 |g|_r}{(r-s)^{1+\sigma}(s-r')}$$

which for $s = \frac{r+r'}{2}$ gives [1] the desired estimate (20) for ψ. This completes the proof of Lemma 9. □

[1] In the last step we used the inequality $(1 - e^{-2\pi q})^{-1} < q^{-1}$, $0 < q < \frac{1}{2}$.

PROOF OF LEMMA 8. Introducing the abbreviations $(h_{12}u_\theta u_\theta^+)^{-1} = p$ and $-u_\theta E(u) = g$, we rewrite (14) as a system, where μ is a constant to be chosen:

$$\begin{cases} \nabla^*\psi = g \\ p^{-1}\nabla w = \psi + \mu. \end{cases} \tag{25}$$

By Lemma 9 (recall that the mean value $[g] = 0$ by (4)) we get a unique ψ with zero mean and satisfying (20):

$$|\psi|_{r'} \leq \frac{c(K,\sigma)}{(r-r')^\tau}|g|_r. \tag{26}$$

for any $0 < r' < r$. We now determine w from the second equation in (25): $\nabla w = p(\psi + \mu)$. Since the left–hand side has mean value zero, we must require the same of the right–hand side:

$$\mu = -\frac{\int p\psi d\theta}{\int p d\theta}. \tag{27}$$

From the above estimates on h_{12} and on u_θ we obtain $|\mu| < \kappa M N^4 c\frac{|g|_r}{(r-r')^\tau}$. We conclude that there exists a unique solution w satisfying

$$|w|_\rho \leq c_1\frac{|p(\psi+\mu)|_{r'}}{(r'-\rho)^\tau} \leq \frac{c_2}{(r'-\rho)^\tau(r-r')^\tau}|g|_r;$$

here (26) and (27) were used in the second inequality. Setting $r' = (r+\rho)/2$ we obtain, recalling that $|u_\theta| < N$:

$$|w|_\rho \leq \frac{c_3}{(r-\rho)^{2\tau}}|E(u)|_r, \quad c_3 = c_3(M,N,K,\kappa,\sigma). \tag{28}$$

Using $|u_\theta| \leq N$ again we get the first estimate of (18). Using the Cauchy estimate $|v_\theta|_\rho \leq \frac{|v|_r}{s-\rho}$ (with appropriate s) one obtains also the second inequality (18). This completes the proof of Lemma 8 and the analysis of the homological equation. □

6. Quadratic dependence of the error

The updated $\tilde{u} = u + v$ $(v = u_\theta w)$ obtained by Newton's iteration is expected, according to our discussion of section 2, to give a quadratic improvement of the error. The following lemma makes this precise:

LEMMA 10. *Let u satisfy the assumptions of Lemma 8 and, moreover, let $\tilde{u} = u + v$ satisfy $(\tilde{u}, \tilde{u}^+) \in \mathcal{D}_R$ for $|Im\,\theta| < \rho$, for some $0 < \rho < r$. Then*

$$|E(\tilde{u})|_\rho \leq \frac{c_6}{(r-\rho)^{4\tau}}|E(u)|_r^2, \tag{29}$$

where $c_6 = c_6(M,N,K,\kappa,\sigma)$.

PROOF. With u as specified, we estimate the error

$$|E(\tilde{u})|_\rho \equiv |E(u+v)|_\rho = |E(u) + E'(u)v + Q|_\rho$$

with Q denoting the remainder. From (13) we obtain

$$u_\theta E(u) + u_\theta E'(u)v = vE'(u)u_\theta,$$

or $E(u) + E'(u)v = wE'(u)u_\theta = w\frac{d}{d\theta}E(u)$. Using (28) for w, together with the Cauchy estimate $|\frac{d}{d\theta}E(u)|_\rho \le c\frac{|E(u)|_r}{r-\rho}$, we obtain

$$|E(u) + E'(u)v|_\rho \le c_4 \frac{|E(u)|_r^2}{(r-\rho)^{2\tau+1}} < c_4 \frac{|E(u)|_r^2}{(r-\rho)^{4\tau}}, \tag{30}$$

where $c_4 = c_4(M, N, K, \kappa, \sigma)$. Moreover, the error Q is quadratically small in terms of $|E(u)|_r$. Indeed, by Taylor's formula we have for some $0 \le t \le 1$:

$$Q = \frac{1}{2}\frac{d^2}{dt^2}E(u + tv).$$

From (3) we have

$$|Q|_\rho \le c_5 |v|_\rho^2, \tag{31}$$

where c_5 depends only on $|h|_{C^3}$. From this, from (18) and from (30) we obtain the desired estimate (29). □

7. The limiting process

In this section we specify the details of iteration of the Newton step (15) and show that it converges to the solution of $E(u) = 0$.

We choose the sequence $r = r_0 > r_1 > \dots$, with $r_n \to r_\infty > 0$, according to $r_n = r_\infty + 2^{-n}(r_0 - r_\infty)$, and construct a sequence $u_0, u_1, \dots, u_n, \dots$ via $u_n = u_{n-1} + (u_{n-1})_\theta w_{n-1}$, where $w_{n-1} \in W_{r_n}$ is the solution of the homological equation (14) with $u = u_{n-1}$.

We show first that the solution of $E(u) = 0$ exists (provided δ is small enough) by repeated application of Lemmas 8 and 10; their applicability will be justified below.

Applying (18) of Lemma 8 to $u = u_n$, $r = r_n$, $N = 2N_0$, we obtain the estimates for corrections $v_n = u_{n+1} - u_n$:

$$|v_n|_{r_{n+1}} \le \frac{c_1}{(r_n - r_{n+1})^{2\tau}}\epsilon_n, \tag{32}$$

where $\epsilon_n = |E(u_n)|_{r_n}$ and where $c_1 = c(M, 2N_0, K, \sigma)$ is the constant from Lemma 8, and

$$|(v_n)_\theta|_{r_{n+1}} \le \frac{c_1}{(r_n - r_{n+1})^{2\tau+1}}\epsilon_n. \tag{33}$$

By Lemma 10, see (29), ϵ_n satisfy

$$\varepsilon_{n+1} \le c_6 \frac{\varepsilon_n^2}{(r_n - r_{n+1})^{4\tau}} = c_7 a^n \epsilon_n^2, \tag{34}$$

where $a = 2^{4\tau}$ and $c_7 = c_6\frac{2^{4\tau}}{(r_0 - r_\infty)^{4\tau}}$. Thus the geometric growth a^n competes with quadratic decay ε_n^2; with $\varepsilon_0 \equiv \delta$ chosen sufficiently small – it is here that the choice of δ in Theorem 6 comes in – the quadratic decay wins: $\varepsilon_n \to 0$. Indeed, rescaling the sequence by introducing

$$\eta_n = a^{n+1} c_7 \varepsilon_n \tag{35}$$

we obtain $\eta_{n+1} \le \eta_n^2$. Therefore, if $\eta_0 = ac_7\varepsilon_0 < 1$ we have $\eta_n \le \eta_0^{2^n} \to 0$ and $\varepsilon_n \to 0$. By (32) we then have $|v_n|_{r_\infty} \to 0$ faster than exponentially, so that $u_\infty = \lim_{n\to\infty} u_n = u_0 + \sum_0^{n-1} v_k$ is well defined in $|\text{Im}\ \theta| < r_\infty$. We conclude: $E(u_\infty) = \lim_{n\to\infty} E(u_n) = 0$.

We prove now that the assumptions (16) and (17) of Lemmas 8 and 10 hold with $u = u_n$, $N = 2N_0$, $r = r_n$ at nth step for all $n \geq 1$. It suffices to show that for all $n \geq 1$

$$|u_n - u_0|_{r_n} < \frac{R}{2} \tag{36}$$

and

$$|(u_n - u_0)_\theta|_{r_n} < \frac{1}{2N_0}, \tag{37}$$

for all $n = 1, 2, \ldots$. Indeed, one verifies that (36) together with (7) implies (16) with $u = u_n$, $r = r_n$, while (37) with (8) imply (17) with $u = u_n$, $N = 2N_0$ and $r = r_n$. Thus the constant c_1 in (32) and (33) is given by $c_1 = c(M, 2N_0, K, \kappa, \sigma)$, where c is from Lemma 8. Similarly, $c_6 = c_6(M, 2N_0, K, \kappa, \sigma)$. It remains thus to prove (36) and (37).

For this purpose we first verify, for $\lambda \leq 4\tau$, hence $2^\lambda \leq a$, the estimate

$$\sum_0^\infty \frac{\varepsilon_n}{(r_n - r_{n+1})^\lambda} \leq c_7^{-1}(r_0 - r_\infty)^{-\lambda} \sum_0^\infty \eta_n \leq \frac{2\eta_0}{c_7(r_0 - r_\infty)^\lambda} = \frac{2a\varepsilon_0}{(r_0 - r_\infty)^\lambda}, \tag{38}$$

for $\eta_0 < 1/2$, where we used $\eta_n < \eta_0^{n+1}$. This sum can be made small by choice of $\varepsilon_0 = \delta$.

To prove (36) and (37) we assume that $n \geq 1$ is the smallest integer for which at least one these inequalities is violated. Then, for $\nu < n$ these inequalities hold and we can use the estimate (32), and with (38) we get

$$|u_n - u_0|_{r_n} < \sum_{\nu=0}^{n-1} |v_\nu|_{r_\nu} \leq \sum_{\nu=0}^{n-1} \frac{c_1 \varepsilon_\nu}{(r_\nu - r_{\nu+1})^{2\tau}} < c_1 2a \frac{\varepsilon_0}{(r_0 - r_\infty)^{2\tau}}.$$

Similarly, we find from (33)

$$|(u_n - u_0)_\theta|_{r_n} < c_1 2a \frac{\varepsilon_0}{(r_0 - r_\infty)^{2\tau+1}}.$$

By further decreasing $\delta = \varepsilon_0$, if necessary, we assure that (36) and (37) hold for all n. This completes the proof of the Main Theorem.

REMARK 11. We have proven along the way that if $\frac{|E(u_0)|_r}{(r-\rho)^{4\tau}} < \delta'$ (with a δ' independent of r, ρ) then there exists a solution $u \in W_\rho$ of $E(u) = 0$ satisfying the estimates

$$|u - u_0|_\rho < c_1 2a \frac{|E(u_0)|_r}{(r - \rho)^{2\tau}}, \quad |(u - u_0)_\theta|_\rho < c_1 2a \frac{|E(u_0)|_r}{(r - \rho)^{2\tau+1}}.$$

8. Appendix

A. Small Twist.

1) Statement of the result. For several applications it is important to treat the case where the "twist" $\frac{\partial x_2}{\partial y_1}$ in formula (10) is small. This degenerate case gives difficulties in the representation of the mapping by a generating function, since this requires solving the equation $x_2 = f(x_1, y_1)$ for y_1. On the other hand this situation is unavoidable as it arises, for example, in the stability problem of an elliptic fixed point of an area-preserving mapping. We will show how these difficulties can be overcome, essentially by proper rescaling and introducing a more suitable generating

function. In the subsequent section we will apply the resulting theorem to the above mentioned stability problem.

We assume that the area-preserving real analytic map φ depends on a parameter $\gamma \in (0,1)$ and has the form

$$\begin{aligned} x_2 &= x_1 + a(\gamma) + \gamma y_1 + f(x_1, y_1, \gamma) \\ y_2 &= y_1 + g(x_1, y_1, \gamma) \end{aligned} \tag{39}$$

where the functions on the right hand side are analytic in the complex disc $|y_1| \le 1$ and in $|Im\ x_1| \le r$. We denote $W_{r,s}$ the space of real-analytic functions, bounded analytic in $|y_1| < s$, $|Im\ x_1| < r$ of period 1 in x_1, with the norm $|f|_{r,s} = sup|f|$ over this domain.

THEOREM 12. *There exists a positive constant δ, independent of $\gamma \in (0,1)$ such that if*

$$|f|_{r,1} + |g|_{r,1} < \gamma\delta$$

then there exists an invariant curve $x_1 = u(\theta), y_1 = v(\theta)$ in $-1 < v(\theta) < 1$ with both $u(\theta) - \theta = \hat{u}(\theta)$ and $v(\theta)$ of period 1, real analytic and with $u_\theta > 0$.

This theorem differs from the previous one as the rotation number ω is not prescribed but has to be constructed in an interval of length γ. We need not impose any smoothness conditions on $a(\gamma)$. As the rule this theorem is used in the following way: If $|f|_{r,1} + |g|_{r,1} = o(\gamma)$ then there exists an invariant curve for sufficiently small γ. (Here we use the Landau notation: $f = o(\gamma^k)$ means $|f|\gamma^{-k} \to 0$ for $\gamma \to 0$, while $f = O(\gamma^k)$ means $|f|\gamma^{-k}$is bounded).

2) Generating function. To construct a generating function for this mapping we do not use x_1, x_2 as independent variables but rather x_1 and the rescaled variable $p = \gamma^{-1}(x_2 - x_1 - a(\gamma))$ so that

$$p = y_1 + \gamma^{-1} f(x_1, y_1, \gamma).$$

Since $\gamma^{-1} f = O(\delta)$ is assumed to be small we can solve this equation for $y_1 = F(x_1, p, \gamma) = p + O(\delta)$ and, inserting this into the second equation of (39), we obtain $y_2 = G(x_1, p, \gamma) = p + O(\delta)$. These functions can be defined for $|p| < \vartheta$, for any $0 < \vartheta < 1$, e.g. for $\vartheta = \frac{3}{4}$ if δ is sufficiently small.

Now we use that $y_2 dx_2 - y_1 dx_1$ is exact, hence also

$$\gamma^{-1}(y_2 dx_2 - y_1 dx_1) = \gamma^{-1}(y_2 - y_1) dx_1 + y_2 dp := dl(x_1, p, \gamma)$$

which defines the generating function $l = l(x_1, p, \gamma)$ up to a constant. Thus we have $y_2 - y_1 = G - F = \gamma l_{x_1}, y_2 = G = l_p$, so that $l(x_1, p) = \frac{1}{2}p^2 + O(\delta)$, where the estimate is understood as being uniform in γ.

If we compare this generating function $l(x_1, p)$ (we suppress the γ) with $h = h(x_1, x_2)$ we find the relation $l(x_1, p) = \gamma^{-1} h(x_1, x_1 + a + \gamma p)$. Note that the domain of definition of h depends on γ while that of l is fixed, say $|p| < \vartheta$.

The function $l(x, p)$ can be viewed as the discrete analogue of the Lagrange function $L(x, \dot{x})$ in mechanics.

3) The difference equation. We express the condition that $x_1 = u(\theta) = \theta + \hat{u}(\theta)$, $y_1 = v(\theta)$ represents an invariant curve with rotation number ω -which is yet to be determined: Notice that the invariance requires that $x_2 = u(\theta + \omega)$, hence

$$p = \gamma^{-1}(u^+ - u - a) = \nabla \hat{u} + \alpha$$

where we defined the rescaled ∇ and α by $\nabla f = \gamma^{-1}(f(\theta+\omega)-f(\theta))$, $\alpha = \gamma^{-1}(\omega - a)$. Similarly we define the rescaled functional $E(u)(=\gamma^{-1}(h_1(u,u^+)+h_2(u^-,u)))$ by

$$E(u) = \gamma^{-1}(l_p - l_p^-) - l_{x_1} = \nabla^* l_p(x_1, \nabla\hat{u}+\alpha) - l_{x_1}(x_1, \nabla\hat{u}+\alpha).$$

One recognizes the equation $E(u) = 0$ as the analogue of the Euler equation in mechanics.

4) Determination of ω. To find a solution of $E(u) = 0$ we have to determine ω so that α lies in the domain of definition of l. It suffices that

$$|\omega - a| < \gamma/2 \text{ so that } -\frac{1}{2} < \alpha < \frac{1}{2}.$$

If we want to find a number ω in this interval satisfying a Diophantine condition we have to replace the factor K in (9) by γK and require that

$$|\omega - \frac{p}{q}| \geq \gamma K q^{-\sigma-2}. \tag{40}$$

One can show by a simple measure theoretical argument that for $\gamma > 0$ there exists a number $\omega = \omega(\gamma)$ satisfying these two requirements (see Siegel-Moser [**15**]).

Having fixed ω we have with $u_0 = \theta$ that $E(u_0) = \nabla^*(\alpha) + O(\delta) = O(\delta)$.

Now we may proceed like in the proof of the main theorem:

THEOREM 13. *With ω chosen as above there exists a $\delta' > 0$ depending on r, σ, K but not on γ such that if $|E(u_0)|_r < \delta'$ there exists a unique solution $u(\theta)$ of $E(u) = 0$ with $\hat{u}(\theta) = u(\theta) - \theta \in W_{r/2}$ and with mean value zero.*

5) Proof. The proof proceeds as before; we just have to verify that the estimates of lemma 9 still hold, uniformly in γ, if we use the rescaled ∇ and the modified Diophantine condition (40). Indeed the small divisors can be estimated independently of γ, as follows. If m is chosen as an integer satisfying $|n\omega - m| \leq \frac{1}{2}$ we have

$$\gamma^{-1}|e^{2\pi i n\omega} - 1| = \gamma^{-1} 2\sin\pi|n\omega - m| \geq \gamma^{-1}4|n\omega - m| \geq 4Kn^{-\sigma-1}.$$

The rest of the proof is straightforward.

B. Stability of elliptic fixed points. An area-preserving real-analytic mapping φ can, in the neighborhood of a fixed point, which we take to be the origin, be written in the form $\varphi : (u,v) \longrightarrow (u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha) + O(u^2+v^2)$, or in complex notation with $w = u + iv$

$$w \longrightarrow we^{i\alpha} + O(|w|^2)$$

If $q\alpha/2\pi$ is not an integer for $q = 1, 2, \ldots, k$ then one can find symplectic coordinates such that the mapping is in Birkhoff normal form up to order $k-1$, i.e.

$$w \longrightarrow we^{i\psi} + O(|w|^k) \tag{41}$$

where ψ is real polynomial of degree $\leq \frac{k}{2} - 1$ in $|w|^2$. Its coefficients are invariant under symplectic transformations; they are the so-called Birkhoff invariants.

THEOREM 14. *If in (41) ψ is not a constant then the origin is a stable fixed point under φ.*

PROOF. We may assume that $\psi = \alpha + \beta|w|^{2m}$ where $\beta \neq 0$ and $2 \leq 2m \leq k-1$. Truncating at the order $2m+2$ we may assume that the mapping has the form

$$w \to we^{i\psi} + O(|w|^{2m+2}),\ \psi = \alpha + \beta|w|^{2m}.$$

Since for φ^{-1} the coefficient β is replaced by $-\beta$ we may also assume $\beta > 0$. The stability will be established by exhibiting invariant curves surrounding the origin in any of its neighborhoods. More specifically we will construct such invariant curves in any annulus

$$\varepsilon^2(1-\varepsilon^\nu) \leq |w|^2 \leq \varepsilon^2(1+\varepsilon^\nu),\ \nu = \frac{1}{3}$$

for sufficiently small $\varepsilon > 0$. Notice that this annulus is rather narrow since its width is of the order $\varepsilon^{1+\nu} = \varepsilon^{4/3}$ which is small compared to the radius ε.

The existence of the invariant curves will be deduced from the theorem of the previous section, by using suitable "polar coordinates" x, y which are defined by

$$w = \varepsilon\sqrt{1+\varepsilon^\nu y}\ e^{2\pi i x}$$

so that the above annulus is given by $-1 \leq y \leq 1$. One verifies that the Jacobian of this coordinate change is a constant, namely $-\pi\varepsilon^{2+\nu}$. Therefore the transformed mapping $(x_1, y_1) \to (x_2, y_2)$ will preserve the area element $dx \wedge dy$ and, in fact be exact symplectic.

To express the mapping in these coordinates we observe that for the image point $w_2 = \varphi(w)$ we have $|w_2|^2 = |w|^2 + O(|w|^{2m+3})$ hence $\varepsilon^2(1+\varepsilon^\nu y_2) = \varepsilon^2(1+\varepsilon^\nu y_1) + O(\varepsilon^{2m+3})$ or $y_2 = y_1 + O(\varepsilon^{2m+1-\nu})$. Similarly we find $x_2 = x_1 + \frac{1}{2\pi}(\alpha + \beta\varepsilon^{2m}(1+\varepsilon^\nu y_1)^m) + O(\varepsilon^{2m+1})$ or, expanding the m^{th} power, we have

$$x_2 = x_1 + a + \gamma y_1 + O(\varepsilon^{2m+2\nu}),\ y_2 = y_1 + O(\varepsilon^{2m+1-\nu})$$

where $a = \frac{1}{2\pi}(\alpha + \beta\varepsilon^{2m}), \gamma = \frac{m\beta}{2\pi}\varepsilon^{2m+\nu} > 0$. Notice that the two error terms have the same order by our choice of ν.

This mapping has the form required for the theorem of the previous section, and since $\varepsilon^{2m+2\nu} = \varepsilon^{2m+1-\nu} = O(\gamma^{1+\mu})$ with $\mu = \frac{1}{6m+1} > 0$ the error term is indeed small compared to γ. Moreover, the mapping is real analytic in $|y_1| < 1, |Im\ x_1| < r$. Thus, for sufficiently small ε, every such annulus contains an invariant curve, proving our claim. □

C. Comments on the literature. The original invariant curve theorem [**8**] was proven under different assumptions: The mapping there was neither real analytic nor exact symplectic but only a sufficiently smooth mapping satisfying a certain curve intersection property (which entered also in G. D. Birkhoff's proof of his extension of the "last geometric theorem of Poincaré" [**1**]. In this sense the present statement is more restrictive but still sufficient for many applications. It should be pointed out that our proof can be extended to cover the case of smooth area-preserving mappings, by employing the approximation technique described in [**11**], [**14**]. The case of real analytic mappings satisfying the curve intersection property has been treated in [**15**], but using transformation theory.

As mentioned in the introduction the present proof originated in the study of analogue theorems for partial differential equations, where a transformation theory is not available. This was carried out in [**9**], where also the trick of the "variation of constants" can be found. For the Hamiltonian case the corresponding proof was carried out by Salamon and Zehnder in [**14**].

There is a large literature on the more delicate aspects of the invariant curve theorem. For example, the minimal smoothness assumption for the validity of the theorem have been studied by Rüssmann [12] and in the extensive papers by M. Herman [3], [4], where one finds a wealth of further results on this problem, e.g on the break-down of invariant curves. In the analytic case the Diophantine condition can be replaced by a weaker one, the so-called Brjuno condition. This situation has been studied with great precision by Rüssmann, see [13]. In particular, we want to point out that his arguments in [12] and [10] lead to an improvement of the estimates in Lemma 9: one can replace the exponent $\sigma + 2$ by $\sigma + 1$ in (20), and consequently one can replace $\tau = \sigma + 2$ by $\tau = \sigma + 1$ in the text following this Lemma.

This list of references is certainly not complete, and the reader can find more information in the papers quoted. In particular, the Section 16 of the survey of Mather and Forni [7] contains a discussion of various results on annulus mappings.

We also want to point out that generically analytic maps possess near an elliptic fixed point besides the invariant curves also invariant Mather sets, which are Cantor sets. More precisely, if one fixes a germ of an elliptic fixed point of general type, one can, by arbitrarily small change of terms of arbitrary higher order, achieve that the so modified map possesses such nondegenerate Mather sets in every neighborhood of the fixed point. This was proven by C. Genecand [2], sharpening the earlier paper by Zehnder [16]. All the above mentioned proofs are based on functional analytic methods and iteration technique. An interesting proof avoiding these iterative approximations entirely is the paper by Y. Katznelson and Ornstein [5], [6]. It is based on delicate estimates on the orbits of the mappings. It gives sharper results, but is not easy to read.

Applying (18) of Lemma 8 to u_0 with $r = r_0$ and $N = N_0$, we obtain

$$|u_1 - u_0|_{r_1} < \frac{R}{4}$$

and

$$|(u_1 - u_0)_\theta| < \frac{1}{2U_0},$$

for small enough δ. The first of these two estimates implies $(u_1, u_1^+) \in D_R$, while the second, together with the bounds (8) on $(u_0)_\theta$, implies $|(u_1)_\theta|\ |(u_1)_\theta^{-1}| < 2U_0$.

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