

What is Entropy?

A short and clear answer to the question in the title of this article may be difficult to find (at least, it was for me). I would therefore like to illustrate the entropy of gas on the simple parody of ideal gas: a single particle—a “molecule”—that bounces elastically between two stationary walls. To strip away the remaining technicalities, let’s take the molecule’s mass as $m=1$. Our system has one degree of freedom $n=1$ (as opposed to $n=3$ in a monatomic gas).

Entropy in One Sentence

The phase plane trajectory of the particle in Figure 1 is a rectangle with area $2vL$. The entropy S of the “gas” in Figure 1 is the logarithm of the area inside the phase plane trajectory:¹

$$S = \ln(\text{area}) = \ln(2vL). \quad (1)$$

Taking the log in (1) is a natural (apologies for the pun) choice. For one, it makes the entropy additive. Indeed, consider a system that consists of two “vessels” with lengths L_1, L_2 and speeds v_1, v_2 (see Figure 2a). The phase space is now the Cartesian product $\{(x_1, v_1; x_2, v_2)\}$. The four-dimensional volume of the Cartesian product of two rectangles is $2v_1L_1 2v_2L_2 = \text{Area}_1 \text{Area}_2$; the logarithm of the product is thus $S = S_1 + S_2$.

Another motivation for taking the logarithm is that it makes entropy (1) analogous to Shannon’s entropy. Roughly speaking and in the simplest setting, this latter entrop-

¹ The same definition applies to the actual gas; in that case, the area is replaced by the volume that is enclosed by the energy surface in the phase space. The dimension of this space is enormous, and the volume must be measured in appropriate units.

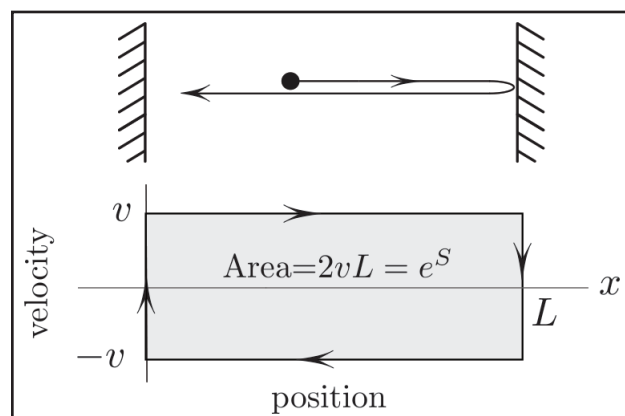


Figure 1. The phase trajectory of a particle that bounces with speed v encloses area $2Lv$.

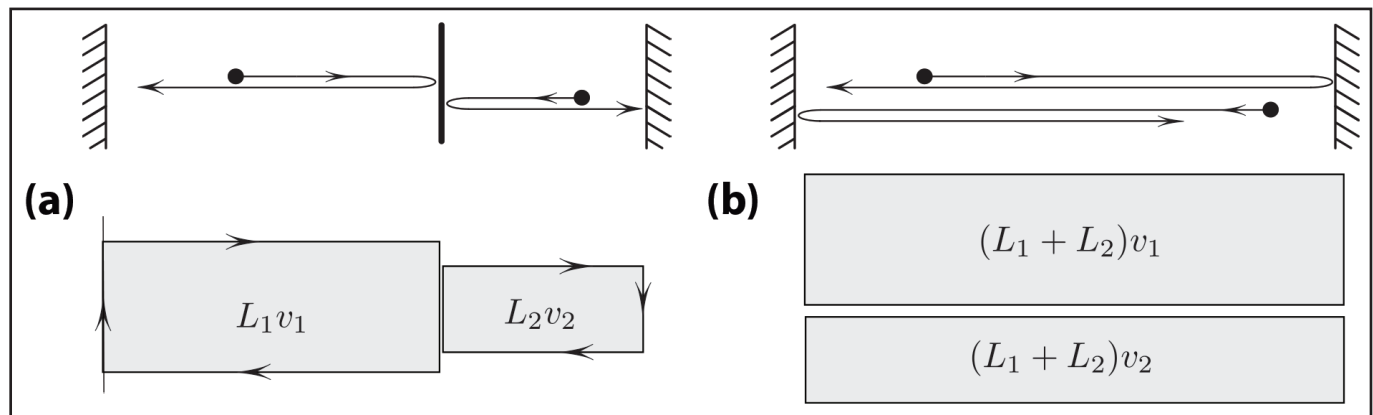


Figure 2. Entropy increases when the wall between the vessels is removed.

py counts the number of letters in a word; this number is then the logarithm of the number of possible words.² The area in Figure 1 is the most natural measure of the “number of possible states,” thus making S a continuous analog of Shannon’s entropy.

If we add small amount dQ of heat to the gas at temperature T , why does the entropy change according to

$$dS = \frac{dQ}{T}, \quad (2)$$

the standard formula in most physics textbooks? Feynman’s beautiful lectures explain (2) in a more realistic setting but take a few pages, so here is a quick explanation of (2) in our simple case.

Explanation of (2)

Let’s add heat dQ by increasing the kinetic energy of the molecule in Figure 1, which amounts to increasing v :

$$dQ = d(v^2/2) = v dv. \quad (3)$$

The change dv in turn causes $S = \ln(2vL)$ to change:

$$dS \stackrel{(1)}{=} \frac{dv}{v} = \frac{v dv}{v^2} \stackrel{(3)}{=} \frac{dQ}{v^2} = \frac{dQ}{T},$$

where the “temperature”

² As long as one allows all possible combinations of letters and each letter is equally probable.

$$T = v^2 \quad (4)$$

is twice the kinetic energy of the molecule. I put “temperature” in quotes because it is measured in units of energy rather than degrees. The coefficient that relates the two units is called the Boltzmann constant.

Boltzmann Constant

One can measure the temperature of a gas in units of average kinetic energy of a molecule³ per degree of freedom, or in Kelvin degrees. The coefficient between these two measures is called the Boltzmann constant k_B :

$$\frac{1}{n} m v^2 = k_B T,$$

where n is the number of degrees of freedom and T is in Kelvin units. To put it differently, k_B is the kinetic energy per molecule per degree of freedom that is required to raise the temperature by $2^\circ K$. Not surprisingly, it is a tiny number: $k_B \approx 1.4 \times 10^{-23} J/K$ (joules per Kelvin). The units of temperature in (4) are the same as those of kinetic energy, so “ k_B ” = 1 (and also $n=1$) in that case.

Entropy Increase

When two substances are mixed, the entropy of the mixture is greater than the sum of the original entropies:

$$S_{\text{mixture}} > S_1 + S_2.$$

For our one-dimensional gas, this statement borders on triviality. Indeed, consider two side-by-side vessels (see Figure 2).

³ Twice the kinetic energy as in (4).

Removing the wall does not affect the speeds but does increase the “volume,” which becomes $L = L_1 + L_2$ so that

$$S_{\text{before}} = \ln(L_1 v_1) + \ln(L_2 v_2) < \ln((L_1 + L_2) v_1) = S_{\text{after}}.$$

Adiabatic Processes

All thermodynamics texts mention that entropy does not change during adiabatic processes.⁴ For our “gas,” this amounts to the following rigorously proved statement: If the length $L = L(\epsilon t)$ changes smoothly over the time interval $0 \leq t \leq 1/\epsilon$, then the product Lv changes by the amount $O(\epsilon)$ if ϵ is sufficiently small.

This basic example captures the essential mechanism and the reason for adiabatic invariance of entropy in ideal gases. However, no rigorous proof of adiabatic invariance exists — even for the simplest system of two “molecules” that are modeled by hard disks bouncing inside a square with slowly moving walls. The gap between what we believe and what we can prove is enormous.

In conclusion, it seems that discussions of entropy—at least on simple phase plane examples like the one here—belong in dynamical systems textbooks (and in physics texts when entropy is mentioned).

The figures in this article were provided by the author.

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⁴ I.e., a slow change of vessel with no heat or mass exchange with the outside.