Slings, Bullets, Blow-up, and Linearity

MATHEMATICAL CURIOSITIES
By Mark Levi

The sling is a simple weapon — essentially a pendulum with two strings instead of one, each of which holds a side of a cradle enclosing a projectile. When one spins the pendulum and releases a string, the discharged projectile can travel over 400 meters. The sling was the world distance champion for millennia until the English long bow, and then the firearm, surpassed it. Speaking of the latter, a bullet shot into water has an unexpectedly short killing range despite its enormous speed.

Both of these phenomena—the surprisingly long killing distance of the sling and the surprisingly short killing distance of a water-bound bullet—involve near-infinite (on the human scale) velocities. Both can be "explained" by the fact that solutions of the simple ordinary differential equation (ODE) $\dot{x}=x^2$ blow up in finite time, something usually presented as a mere scholastic curiosity.

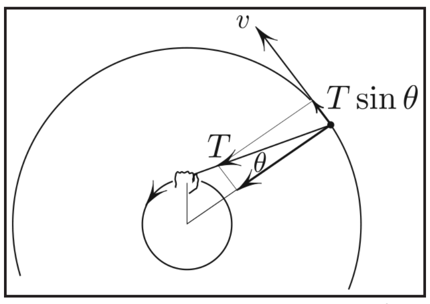

Figure 1. *The velocity v blows up in finite time $T=$ the tension of the string, giving rise to the accelerations $a_{\text{tangential}}$ and $a_{centripetal}$.*

Launching the Sling

The caricature of the sling in Figure 1 shows the hand moving in a circle *so as to make the cradle describe a concentric circle* (to achieve this, the hand must accelerate a certain way). I claim that the speed v then satisfies the ODE mentioned above: $\dot{v}=kv^2$, with a constant k. Indeed, Figure 1 gives us $a_{\text{tangential}}=a_{centripetal}\tan\theta$; and since $a_{\text{tangential}}=\dot{v}$ and $a_{centripetal}=v^2/R$, this yields

$$\dot{v}=kv^2, \quad k=\tan\theta/R.$$

And the solution

$$v=\frac{v_0}{1-v_0kt}$$

does in fact approach infinity as $t\to t_\infty=1/v_0k$ in the idealized non-relativistic world of infinitely fast hands and infinitely strong strings. The "shadow" of this infinite speed is seen in the fact that a good slinger can launch at over $1/5$th the speed of sound (computed from the aforementioned $>400m$ distance. In fact, the actual speed is higher, given that the calculation ignores the air resistance).

Shooting into the Water

Continuing with the weapons-related theme, let us ask: what depth renders harmless a bullet shot down into the water? Assume that (i) the water drag on the bullet is proportional to the square of the speed; (ii) the terminal sinking velocity of the bullet is $1m/sec$, and (iii) the safe velocity of the bullet is $v_s=10m/sec$ (the speed gained in dropping 15 feet, painful but probably not dangerous). With these assumptions, the safe depth turns out to be

$$x_s\approx 0.1\ln\Big(1+\frac{v_0}{v_s}\Big), \qquad (1)$$

in meters. Before deriving this expression, let us find safe depths for various bullet speeds. For $v_0=10^3m/sec$ (about three times the speed of sound), $x_s\approx 46cm$. For the escape velocity $v_0\approx 11km/sec$, $x_s\approx 69\,cm$. Every increase in the order of magnitude simply adds about $23cm$ to the safety depth. And for a bullet travelling at the speed of light (here I am abandoning the last touch with reality), the safe depth is just $2m$! This drastic loss of speed of the bullet is identical, up to the time-reversal, to the drastic gain of the speed of the sling projectile.

Derivation of (1)

The bullet shot straight down obeys Newton's law: $m\dot{v}=-cv^2+mg$, or

$$\dot{v}=-kv^2+g, \quad k=\frac{c}{m}, \qquad (2)$$

where $v=\dot{x}$ and the x-axis points down. From the terminal velocity condition $-kv_{\text{term}}^2+g=0$, we find $k=g/v_{\text{term}}^2\approx 10m^{-1}$. Neglecting g in (2), we get

$$\dot{v}=-kv^2, \quad k\approx 10m^{-1}.$$

Substituting the initial condition $v(0)=v_0$ and the time t_s of reaching velocity v_s into the solution of this ODE gives

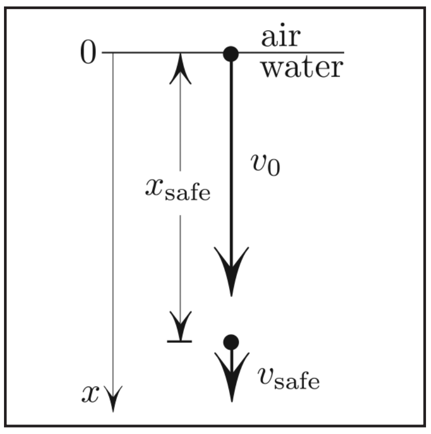

Figure 2. *Shooting into water.*

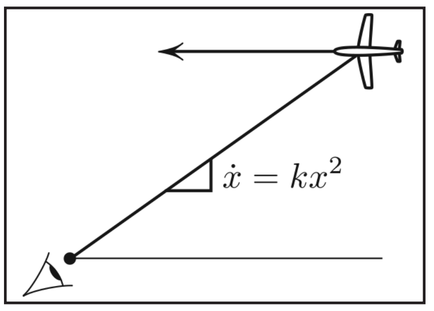

Figure 3. *Geometry of the finite-time blow-up. The slope x of the line of sight satisfies the "sling" ODE.*

$$v_s=\frac{v_0}{1+kv_0t_s}\Rightarrow t_s=k^{-1}(v_s^{-1}-v_0^{-1})<\frac{1}{kv_s}.$$

Integrating v gives $x=k^{-1}\ln(1+kv_0t)$, and substituting $t_s<1/kv_s$ into this expression gives (1).

The Hidden Linearity

The ODE $\dot{x}=x^2$ (a special case of the Riccati equation) hides linear growth, which can be expressed in two equivalent ways. Algebraically, the equation simply amounts to the linear growth of $1/x$, namely to $\frac{d}{dt}\frac{1}{x}=1$. Geometrically, this ODE governs the evolution of the slope $x=v/u$ of solution vectors in the (u,v)-plane of the trivial system $\dot{u}=-1$, $\dot{v}=0$. Figure 3 offers an essentially equivalent realization. The blow-up of the solution occurs when the plane is overhead.

As a concluding remark, the assumption $dv/dt=-kv^2$ amounts to a more natural-sounding statement: the kinetic energy E decays exponentially with the distance $dE/dx=-2kE$.

The figures in this article were provided by the author.

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