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Geometric Phases in the Motion of Rigid Bodies

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Abstract

In this paper we make some differential-geometric observations on the kinematics of convex surfaces rolling along a fixed plane in $\mathbb{R}^3$, and on the relationship of the problem with parallel transport and the Gauss-Bonnet formula. These ideas are then applied to recover "Berry's phase" of a free rigid body which was found by MONTGOMERY using Stokes' Theorem. We also point out a new "twist" on this problem. As a second application, we give a solution of a problem posed by R. BROCKETT. As a third application, we give a geometrical description of "Berry's phase" in $SO(3)$; this can be applied to various rigid-elastic systems to compute their geometric phases.

Contents

1. Rolling Surfaces, Curvatures and Parallel Transport

In this main section we make two observations (Theorems 1 and 2) on the (non-holonomically) constrained motion of convex surfaces in $\mathbb{R}^3$. We then apply these observations to three different problems: the free motion of a rigid body, a problem of rolling a ball on the plane proposed by R. BROCKETT and finally to the motion of a rigid body with a rotor, analyzed recently in [BKMS].

Theorem 1. *Consider an arbitrary smooth solid convex* surface S, and let C_S be a smooth closed directed curve on S; for simplicity assume that C_S has no self-intersections (see Fig. 1). Consider a rigid motion of S in the upper half-space $z \geqq 0$ in which it rolls without sliding** along the (x, y)-plane so that the contact point traces out the closed curve C_S precisely once in the direction of the given orientation. Then the surface S undergoes a composition of a parallel translation (in the horizontal direction) and a rotation through some angle around the z-axis. This angle is given by any of the following three expressions:*

$$\alpha = \int_0^L k(C_S)(s)\, ds + \int_0^T \omega_z(t)\, dt - 2\pi \tag{1.1a}$$

$$= -\int\int_{\mathrm{Int}\,C_S} K\, d\sigma + \int_0^T \omega_z(t)\, dt \tag{1.1b}$$

$$= -\Omega + \int_0^T \omega_z(t)\, dt, \tag{1.1c}$$

*where the geodesic curvature $k(C_S)(s)$ is defined with respect to the orientation on the surface S induced by the orientation of the xy-plane***, s is the arc-length, ω_z is the component of the angular velocity of the surface (with the parameter t treated as time) in the z-direction, Int C_S is the disk on S bounded by C_S, i.e., that disk on S which lies to the left of C_S when it is traversed in the positive direction ("left" is defined as the cross product of the inward normal to S and the tangent to the curve), K is the Gaussian curvature of the surface S, and $\Omega \equiv \int\int_{\mathrm{Int}\,C_S} K\, d\sigma$ is the solid angle of the image of Int C_S under the Gauss map, i.e., the positive solid angle of the cone of normal vectors at the points of Int C_S (with all vectors carried to the origin).*

The proof of this theorem, as well as of those to follow, is given in Section 3.

One might venture a naive and incorrect guess that $\alpha = \int_0^T \omega_z(t)\, dt$ – it seems reasonable to say that the angle is the integral of the angular velocity! The correction term is an example of what is often called the geometric phase, or the Berry phase, other examples of which together with further discussion can be found in [BH, B, C, GKM, H, KS, L, MMR, S].

* We assume that the Gaussian curvature of the surface is positive; we eschew generality to minimize the technicalities.

** The precise meaning of "rolling without sliding" is this: 1) the surface is tangent to the plane at all times, and 2) the point of the surface which is in contact with the plane has zero speed at the moment of contact. A caution: the angular velocity is allowed to have a vertical component, as in the case shown in Fig. 2 below. A familiar example is a rolling coin: if it does not go in a straight line, then its angular velocity has a vertical component. Another example of rolling without sliding is that of a ball spun around its vertical axis and placed on the horizontal plane which subsequently is slightly inclined (no slippage occurs). The ball slowly travels along, spinning around a nearly vertical axis. Again, such motion satisfies the definition of rolling without sliding.

*** This amounts to defining the the orientation of S with respect to the *inward normal* to S, contrary to the more usual way of using the outward normal.

What is the relationship between the two contact curves C_S and C_P? The answer is given by the following theorem, which is essentially the infinitesimal version of Theorem 1.

Theorem 2. *In the notations of the previous theorem, let C_P be the curve traced out on the (x, y)-plane by the rolling surface, with the positive direction along both curves C_S and C_P given by the motion of the point of contact* (cf. Fig. 1). *Then*

$$k(C_P) = k(C_S) + \frac{\omega_z}{v} \tag{1.2}$$

where ω_z is the z-component of the angular velocity and $v > 0$ is the speed of the point of contact.

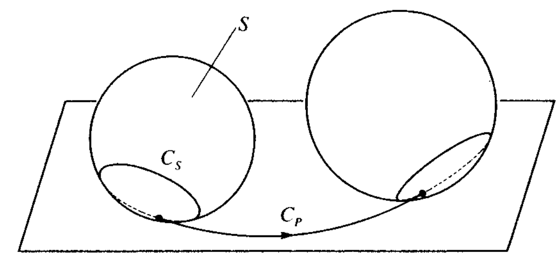

Fig. 1. Rigid convex surface S rolling along the plane.

The sign of both curvatures in this theorem is defined relative to the usual orientation of the (x, y)-plane in the (x, y, z)-space which coincides with that given by the inward normal to S.

The proof of this theorem is based on the following observation about *planar* curves: Consider two curves A and B in the plane, and let us subject the curve B to a rigid motion in such a way that it stays tangent to A at all times and that the point of B which is tangent to A has zero velocity at all times – in other words, B rolls along A without sliding. Then $k_A = k_B + \omega/v$, where ω is the angular velocity of B and v is the velocity of the point of contact. Full details are given in Section 4.

It may seem strange that the velocities occur in the geometric statement (1.2), until one observes that they occur in a ratio. A more geometrical way to state (1.2) would be to choose the arc-length along C_P as time, i.e., to take $v = 1$.

2. Applications

A. A kinematic definition of geodesics

Corollary 1. *If C_P is a straight line and if $\omega_z = 0$, then C_S is a geodesic.*

The proof is an immediate consequence of (1.2).

B. Curvatures and the dynamics of free rigid bodies

It is well known from the work of Poinsot (cf. [A]) that when a rigid body is moving without external forces, its ellipsoid of inertia rolls without sliding

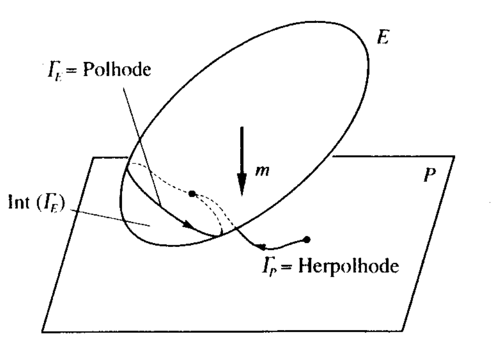

Fig. 2. The rigid body dynamics in the Poinsot description. The moments of inertia around the three axes of the ellipsoid are I_1, I_2, I_3. The energy E and the angular momentum vector m are conserved quantities. The ellipsoid of inertia rolls without sliding along the plane, with the arrows showing the direction of motion of the point of contact.

along a certain plane fixed in space*; see Fig. 2 and the Appendix for more details. Applying Theorem 2 to this situation we obtain

Corollary 2. *The curvatures of the polhode Γ_E (the curve on the ellipsoid of inertia of the body) and the herpolhode Γ_P (the curve in the plane) of Fig. 2 are related by*

$$k(\Gamma_P) = k(\Gamma_E) - \omega_m/v, \tag{2.1}$$

where ω_m is the component of ω in the direction of the angular momentum vector and is constant $\omega_m = 2E/\mu$ (see Appendix), where $\mu = |m|$ and where E is the energy. (Recall that $k(\Gamma_E)$ is measured with respect to the orientation of the inward *normal to the ellipsoid E, dictated by the orientation of the plane.)*

The proof of the corollary follows at once from (1.2) and from the fact that $\omega_z = -\omega_m$, Fig. 2.

C. *Berry's phase in the free motion of rigid bodies*

As in the previous section we consider a free motion of a rigid body. The position of the body in space repeats itself periodically modulo rotation around the axis of the angular momentum m; this repetition happens each time the polhode rolls out on the plane exactly once. Let α denote the angle between the initial and the final positions of the body; from Fig. 2 it is clear that $\alpha = \int_0^L k_P\, ds \pmod{2\pi}$, where k_P is the curvature of the polhode (i.e., of the planar track left by the ellipsoid) and L is its length.

From (2.1) we obtain at once that $\alpha = \int_0^L k_E - \int_0^T (\omega_m/v)\, dt \pmod{2\pi}$. Since $\omega_m = \text{const}$ (see the Appendix), and since $\int_0^T (ds/v) = \int_0^T dt = T$, we obtain for the angle $\phi = -\alpha$ corresponding to the turn around the m-axis

* The relevant background on the rigid body dynamics is given in the Appendix.

Corollary 3. *During one period T of precession, a free rigid body turns around the direction of its angular momentum m through the angle*

$$\phi = -\int_0^L k_E\, ds + \omega_m T \pmod{2\pi} = \mp \left| \int_0^L k_E\, ds \right| + \omega_m T \pmod{2\pi}, \tag{2.2}$$

with the "$-$" when $E > E_2$ and with the "$+$" when $E < E_2$, where $E_2 = \mu^2/2I_2$ is the energy of rotation around the intermediate axis of inertia, and $\mu = |m|$. The sign of k_E is computed with respect to the direction of the motion of the point of contact, using the orientation of the ellipsoid given by the inward *normal.*

Remark 2. As an interesting consequence of this formula, Berry's phase $\mp \int k_E$ *enhances* the net rotation (modulo 2π) for the range of higher energies while it *retards* it for the range of lower energies; furthermore, it has a discontinuity at the critical value of $E = E_2$. For the critical case $E = E_2 \equiv \mu^2/2I_2$ the period blows up: $T = \infty$, and the formula becomes meaningless. Finally, we observe that the holonomy effect A is largest near the critical values of energy, $E = E_2$.

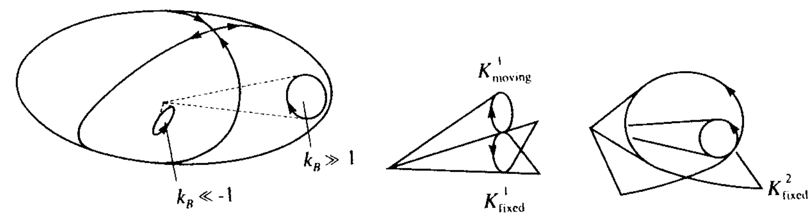

Fig. 3. The rolling cones illustrating the different effects of Berry's phase.

We now explain the reason for the sign $\mp$ in (2.2); another explanation is given in Section 4, in the proof of Corollary 4. Let us consider two cones, K_{fixed} and K_{moving}, one fixed in space and the other in the body, defined as follows (cf. Fig. 3): K_{fixed} is traced out by the rays originating at the center of the ellipsoid and passing through the herpolhode Γ_P, while the cone K_{moving} has the same vertex and is traced out by the rays passing through the polhode Γ_E on the ellipsoid of inertia. By the Poinsot description, K_{moving} rolls without sliding along K_{fixed} and at each moment the two cones share a generator, namely the angular velocity vector. The reason for $\mp$ is now clear from Fig. 3: for $E \approx E_{\max} = \mu^2/2I_1$, the cone K_{moving} does not enclose K_{fixed}, while for $E \approx E_{\min} = \mu^2/2I_3$ it does. This can be justified more rigorously by using (1.2), which for $E \approx E_{\max}$ gives $k_P < 0 < k_E$, while for $E \approx E_{\min}$ it gives $k_P < k_E < 0$, as can be seen from (1.2) with a little extra argument. These inequalities show that the total geodesic curvature of the polhode is negative or positive depending on the sign of $E - E_2$. A simple rigorous way to determine the sign is to apply the Gauss-Bonnet Theorem to the region of the ellipsoid enclosed by the polhode – this is done at the end of Section 4.

The previous corollary will be used to give a new proof of the formula discovered by MONTGOMERY [MMR, M1]:

$$\phi = -(\text{solid angle swept out by } M(t)) + \frac{2ET}{\mu} \bmod 2\pi, \qquad (2.3)$$

where $M(t)$ is the vector of angular momentum expressed in the body frame (see the Appendix), $\mu = |m|$ and E is the energy. More precisely, let $A = \min(\Omega, 4\pi - \Omega)$, where Ω is the solid angle of any of the two cones swept out by the vector $M(t)$ of the body angular momentum; then $0 \leqq A \leqq 2\pi$.

Corollary 4. *For the free motion of a rigid body, let $0 < A < 2\pi$ be the solid angle swept out by the vector $M(t)$ of the angular momentum expressed in the body frame. Then during one period of $M(t)$ the body turns around the direction of the angular momentum m in space through the angle*

$$\phi = \pm A + (2ET/\mu) \pmod{2\pi}, \qquad (2.4)$$

with a "+" for $E > \mu^2/2I_2$ and with a "−" for $E < \mu^2/2I_2$.

Proof of this formula is given in Section 4; it is an immediate result of applying the Gauss-Bonnet Theorem to the ellipsoid of inertia in (2.2).

Remark 3. We note that the expression $2E/\mu$ in MONTGOMERY's formula (2.3) is precisely the component of the angular velocity in the direction of the angular momentum: $\omega_m = \text{proj}_m \omega = (m, \omega)/|m| = (I\omega, \omega)/\mu = 2E/\mu$.

Remark 4. Let k_M denote the geodesic curvature of the curve $M(t)$ on the momentum sphere. Equation (2.2) can be rewritten as

$$\phi = -\int_0^L k_M\, ds + \omega_m T \pmod{2\pi} = \mp \left| \int_0^L k_M\, ds \right| + \omega_m T \pmod{2\pi}, \qquad (2.5)$$

with the same sign convention as in Corollary 3. This is a consequence of the invariance of the geodesic curvature under the Gauss map [L], once we observe that *the curve $M(t)$ is the image of the polhode Γ_E under the Gauss map* (up to a normalization); see the Appendix.

D. Brockett's example

Theorem 2 can be applied to solve a problem suggested by R. BROCKETT: a rigid sphere lies on the table with a horizontal plane resting on the top of the sphere. This plane executes a horizontal circular motion – more precisely, each point of the plane describes precisely one horizontal circle of radius R. There is no sliding at either of the two contact points. What is the new position and the orientation of the sphere?

Applying Theorem 2 we obtain the following answer:

Corollary 5. *As the result of the motion just described (see Fig. 4), the sphere is rotated around its center by the matrix $\Phi \in SO(3)$ given by*

$$\Phi = \exp(\alpha \exp(-\beta\hat{v}_0)\, e_z)^\wedge, \quad \alpha = 2\pi\sqrt{1 + (R/2)^2}, \quad \tan\beta = R/2,$$

where $\hat{}$ is the isomorphism taking a vector $a \in \mathbb{R}^3$ into the matrix $\hat{a}$ in the Lie algebra $so(3)$ according to the rule

$$a \in \mathbb{R}^3 \mapsto \hat{a} \in so(3) \equiv \text{Alt}(3): \ \hat{a}v = a \wedge v, \qquad (2.6)$$

with $\wedge \equiv \times$ denoting the standard cross product.

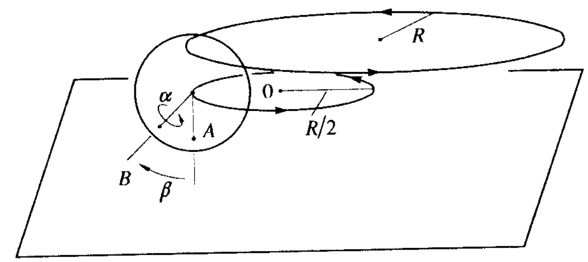

Fig. 4. BROCKETT's example: the sphere undergoes the rotation around the axis B through the angle α where B lies in the vertical plane through O and A and $\angle(B, A) = 0$.

3. Paths in $SO(3)$ and Ribbons in $\mathbb{R}^3$

It is natural to ask: what is special about the rigid motions generated by rolling surfaces, as in Theorems 1 and 2? For the case of rigid motions in $SO(3)$ that answer is: essentially nothing; this is made precise in the theorem below. For the general Euclidean motion given by $E(t)\, v = a(t) + X(t)\, v$, where $X(t) \in SO(3)$ and $a(t) \in \mathbb{R}^3$, to be generated by a surface rolling along a plane fixed in space, one has to require that $\dot{a} \in \text{Ran}\, \dot{X}$, with some additional mild differentiability assumptions. (Here Ran denotes the range. The restriction amounts to the requirement that at every instant the translation be orthogonal to the angular velocity.) In particular, if the motion fixes the origin, i.e., if $E(t) \in SO(3)$, the above condition holds trivially: $a \equiv 0$, and we have Theorem 3 below. Before stating this theorem, we introduce notations and definitions.

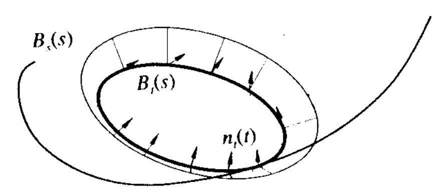

Fig. 5. The ribbon corresponding to a path in $SO(3)$.

Definitions. A *ribbon* is a bundle of normal vectors $n(s)$ along a curve $C: r(s),\ 0 \leqq s \leqq S$ in $\mathbb{R}^3$: $r' \cdot n = 0$. A ribbon *rolls without sliding* along the

plane $z = 1$ with the flow given by $X(t) \in SO(3)$ if the "moved" ribbon $(r_t(s), n_t(s)) \equiv (X(t)\, r(s), X(t)\, n(s))$ satisfies the following two conditions.
1) $r_t(s)$ is tangent to the plane $z = 1$ for all t, i.e., for all $t \in [0, T]$ there exists $s = s(t)$ such that

$$\mathrm{proj}_z\, r_t(s)|_{s=s(t)} = 1,$$

$$\mathrm{proj}_z\, (\partial/\partial s)\, r_t(s)|_{s=s(t)} = 0.$$

2) $$n_t(s)|_{s=s(t)} = e_z \equiv (0, 0, 1).$$

The *geodesic curvature of a ribbon* $(r(s), n(s))$ is defined as the curvature of the projection of the base curve r onto the plane normal to $n(s)$, with the orientation of the plane determined by the direction of n.

Theorem 3. *Consider a curve $X(t) \in SO(3)$ with $X(0) = I$, such that $\omega(t) \cdot e_z > 0$ for all $0 \leqq t \leqq T$, where $\omega \in \mathbb{R}^3$ is the angular velocity (in space) defined by* $\hat{\omega} = \dot{X}X^{-1}$. Then there exists a ribbon $(B_0(\tau), n_0(\tau))$ which is rolled by $X(t)$ without sliding along the plane $z = 1$. Furthermore, the geodesic curvature of the ribbon is given by*

$$k_r = k_p - \omega_z/v, \tag{3.1}$$

where k_p is the curvature of the curve traced in the $z = 1$ plane by the point of contact, v is the speed of that point and ω_z is the z-component of ω. If, moreover, the motion satisfies the periodicity conditions (the first of which simply amounts to $\omega_{\text{body}}(0) = \omega_{\text{body}}(T)$):

$$X^{-1}\dot{X}|_{t=0} = X^{-1}\dot{X}|_{t=T} \quad \textit{and} \quad X(T)\, e_z = e_z, \quad \textit{where } e_z = (0, 0, 1),$$

then the ribbon is closed: $B_0(0) = B_0(T)$ and $n_0(0) = n_0(T)$. Finally, the total geodesic curvature of the ribbon is related to the solid angle Ω enclosed by the cone of normal vectors n all transported to the origin by

$$\int k_r + \Omega = 2\pi; \tag{3.2}$$

to be more precise, Ω is defined as the area on the unit sphere lying to the left of the curve $n(\tau)$, $0 \leqq r \leqq T$, with the orientation of the sphere given by n.

This theorem can be applied to compute the phase in cases when no rolling surface is available a priori, as is the case of a rigid body with a rotor considered by BLOCH, KRISHNAPRASAD, MARSDEN & SANCHEZ-ALVAREZ [BKMS]. Theorem 3 explains geometrically the appearance of the average angular velocity and of the solid angle in [BKMS]. In fact, as long as the conditions on $X(t)$ are satisfied, we can apply Theorem 1 to the ribbon, rather than to the surface, and thus the phase is simply given by (1.1).

Remark. The existence of the base curve $B_0(\tau)$ is equivalent to a theorem of POINSOT on moving cones [W]; the remaining part of the statement on the ribbon and on the geodesic curvatures seems to be new.

* The definition of ^ was given in (2.6).

4. Proofs

To prove Theorem 1 it suffices to prove that (1.1b) holds since (1.1a) follows by the Gauss-Bonnet formula [D], and (1.1c) follows by the definition of the Gaussian curvature. We actually give two proofs of (1.1).

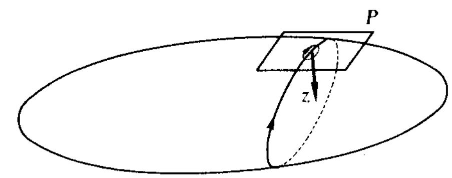

Fig. 6. The use of parallel transport to prove Theorem 1.

First proof of Theorem 1. This proof is based on the idea of parallel transport (see Fig. 6). Let us put ourselves in the body coordinate system and consider the motion of the (x, y)-plane. During this motion the plane stays tangent to the surface S while the tangency point describes the closed loop C_S. The (x, y)-frame defining P has the angular velocity $-\omega_z$ around the normal direction of the z-axis. We start with the particular case of $\omega_z = 0$ and observe that for it the (x, y)-frame in P *undergoes the parallel transport* by the definition of the latter; this is the key idea of the proof. After one trip around the contact curve C_S, the plane P coincides with its initial position while the (x, y)-frame in P makes an angle ψ relative to its position at the outset. This angle, modulo 2π, is precisely the integral of the Gaussian curvature of the region on the surface S enclosed by the curve C_S: $\psi = \iint_{\mathrm{Int}C_S} K\, d\sigma$ (cf. [D]), where the orientation in P induced by the (x, y, z)-frame is used. Now, if $\omega_z \neq 0$, one simply has to add $-\int_0^T \omega_z\, dt$ to obtain $\psi = \iint_{\mathrm{Int}C_S} K\, d\sigma + \int_0^T (-\omega_z)\, dt$, since the parallel transport is an isometry between the tangent spaces. This gives (1.1b) once we observe that $\alpha = -\psi$. □

Second proof of Theorem 1. It suffices to prove (1.1a). The angle α can be computed as the net turn of the tangent vector to the planar contact curve C_P (cf. Fig. 1):

$$\alpha = \int_0^T k(C_P)(t)\, v(t)\, dt \overset{(1.2)}{=} \int_0^T k(C_S)\, v\, dt + \int \omega_z(t)\, dt \pmod{2\pi}.$$

(This defines α modulo 2π; actually, it is reasonable to choose a particular representative $\alpha = \int_0^T k(C_S)\, v\, dt + \int \omega_z(t)\, dt - 2\pi$ because this choice gives $\alpha \approx 0$ in the case when C_S is shrunk to a small neighborhood of a point on S and $\omega_z = 0$. Choosing small α in this case is reasonable because during the entire period $0 < t < T$ the body moves very little from its initial position.) □

We now turn to the proof of Theorem 2, which follows at once from the following two observations.

A. *Let the rigid surface curve be projected onto the (x, y)-plane; in a neighborhood of the tangency point, the projected planar curve undergoes a nearly rigid rotation around the tangency point with the angular velocity ω_z; see Fig. 7.*

B. *Consider two curves confined to the (x, y)-plane, one stationary and the other undergoing the rigid motion of rolling without sliding along the stationary curve. Then the curvatures of the two curves at the corresponding points are related by $k_{\text{stationary}} = k_{\text{moving}} + \omega_z/v$, where ω_z is the angular velocity of the moving curve and v is the speed of the point of contact.*

Observation (A) is heuristically justified by noting that the projection is a near-isometry in the neighborhood of the tangency point.

Observation (B) is heuristically justified by the following argument. Referring to Fig. 7, we consider two consecutive positions of the rolling curve. Let a, b, c denote the angles formed by the tangents to the appropriate curves at the points A, B, C with the x-axis. Here A is the initial point of contact, B is the point of contact at time ds later and C is the new position of the point A at time $s + ds$. Using the definition of curvatures, we have $b - a = k_{\text{stationary}}(s)\, ds$, $b - c = k_{\text{moving}}(s)\, ds$, where ds is the same length in both formulas because of the no-sliding condition. In time ds the tangent at A turns through the angle $\omega_z\, ds$, so that $c - a = \omega_z\, dt = \omega_z\, ds/v$. Subtracting the second of these identities from the first and substituting the last, we obtain the desired statement: $k_{\text{stationary}} - k_{\text{moving}} = \omega_z/v$.

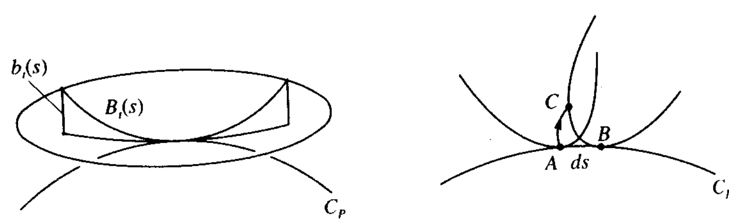

Fig. 7. Proof of the theorem on rolling bodies.

We now set up the machinery for the proof of Theorem 2. Let $s \mapsto B_0(s) \in \mathbb{R}^3$ be the arc-length parametrization of the surface curve at the moment $t = 0$. As the surface rolls along the plane, the curve $C_S = C_S(t)$ undergoes a rigid motion; at time t the point $B_0(s)$ finds itself in a new position $B_t(s) = A(t)\, B_0(s)$, where $A(t)$ is a Euclidean motion acting on $\mathbb{R}^3$. The no-sliding condition is expressed by

$$\partial_t B_t(s)|_{t=f(s)} = 0, \tag{4.1}$$

where the time $t = f(s)$ is defined by the condition of contact and tangency: the point $B_{f(s)}s$ and the tangent vector $\partial_s B_t(s)$ both lie in the (x, y)-plane. Denoting the point of contact by $p(s) = B_{f(s)}(s)$, we obtain a parametrization of the track C_P left by the rolling surface. We note that *s measures the arc-length of C_P as well.* To check this intuitively obvious statement we differentiate $p(s)$ and use the no-sliding condition (4.1):

$$p'(s) = \partial_t B_t(s)\, f'(s) + \partial_s B_t(s) = \partial_s B_t(s), \qquad \text{where } t = f(s),$$

concluding that p' is of unit length. In addition we have just recovered another intuitively evident result: *the two curves C_S and C_P are tangent to each other.* We now rescale the time t so as to have the point of contact move with unit speed, i.e., we define the new time τ by $t = f(\tau)$; now replacing τ with t, we obtain the normalized parametrization $B_t(s)$ where the point of contact now is given by $t = s$.

We record one more intuitively obvious fact as

Lemma 4.1. *At each instant, the surface curve $B_t(\cdot)$ undergoes an infinitesimal rotation around the straight line through the point of contact parallel to $\omega(t)$:*

$$\partial_t B_t(s) = \omega \times (B_t(s) - B_t(t)). \tag{4.2}$$

Proof. Differentiating $B_t(s) = A(t)\, B_0(s)$, where A acts on vectors in $\mathbb{R}^3$ by $A(t)\, v = R(t)\, v + r(t)$, $R \in SO(3)$ and $r(t) \in \mathbb{R}^3$, we obtain

$$\partial_t B_t(s) = \partial_t[R(t)\, B_0(s) + r] = (\dot{R}R^{-1})\, RB_0(s) + \dot{r}.$$

Denoting the skew-symmetric matrix $\dot{R}R^{-1}$ by Ω, we obtain*

$$\partial_t B_t(s) = \Omega[B_t(s) - B_t(t)] + \Omega B_t(t) + \dot{r}. \tag{4.3}$$

Setting $s = t$ which corresponds to the point of contact and using the no-sliding condition (4.1), we get $0 = \Omega B_t(t) + \dot{r}$, and thus $\partial_t B_t(s) = \Omega[B_t(s) - B_t(t)]$. Denoting the entries of Ω by

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_x & \omega_z & 0 \end{pmatrix},$$

we can rewrite (4.3) as (4.2), where $\omega = (\omega_x, \omega_y, \omega_z)$. □

Proof of Theorem 2. Let $b_t(s)$ be the orthogonal projection of $B_t(s)$ onto the (x, y)-plane. The geodesic curvature of $B_t(s)$ at the contact point $B_t(t)$ can be defined as the curvature of the projected curve $b_t(s)$ at $s = t$. To write the formulas for planar curvature it is most convenient to use complex numbers. We recall that the curvature of a planar curve $w(s) = x(s) + iy(s)$ is given by $k_w(s_0) = w''(s_0)/iw'(s_0)$, *provided* $|w'(s_0)| = 1$ and $|w'(s)|'_{s=s_0} = 0$. The last two conditions hold for the planar track C_P since s measures its arc-length, and thus

$$k_P(s) = \frac{p''(s)}{ip'(s)}. \tag{4.4}$$

* $\Omega = \hat{\omega}$, where ω is the angular velocity in the inertial frame.

Similarly, the geodesic curvature of C_S is given by

$$k_B(s) = \frac{\partial_{ss} b_t(s)}{i\partial_s b_t(s)}\bigg|_{s=t}, \qquad (4.5)$$

since $|\partial_s b_t(s)|_{s=t} = 1$ and $\partial_s |\partial_s b_t(s)|_{s=t} = 0$. Indeed, the first relationsip follows from the fact that the curves B and b are tangent while the second holds because the length $|\partial_s b_t(s)| \leq 1$ (since $\partial_s b_t(s)$ is a projection of $\partial_s B_t(s)$) and it is maximized at $s = t$.

To relate the two curvatures in (4.4) and in (4.5), we differentiate $p(s) = b_s(s)$ and use $\partial_t B_t(s)_{s=t} = 0$ after the first differentiation to obtain

$$p''(s) = \partial_{ss} b_t(s)|_{t=s} + \partial_{st} b_t(s)|_{t=s}.$$

Substituting this in (4.4) and using $p' = \partial_s b_t(s)|_{t=s}$ we obtain

$$k_P(s) = \frac{\partial_{ss} b_t(s)}{i\partial_s b_t(s)} + \frac{\partial_{st} b_t(s)}{i\partial_s b_t(s)} \overset{(4.5)}{=} k_S + \frac{\partial_{st} b_t(s)}{i\partial_s b_t(s)}, \qquad t = s.$$

To simplify the last term we differentiate (4.2) with respect to s:

$$\partial_{ts} B_t(s) = \omega \times \partial_s B_t(s),$$

and project this relation onto the (x, y)-plane, obtaining $\partial_{ts} b_t(s) = i\omega_z \partial_s b_t(s)$. □

Proof of Theorem 3. We define the isomorphism $\check{}$ as the inverse of $\hat{}$, i.e., for any skew-symmetric real 3×3 matrix A we define $\check{A} \in \mathbb{R}^3$ via

$$Av = \check{A} \times v. \qquad (4.6)$$

Given the curve $X(t)$ in $SO(3)$, we define the ribbon $(B_0(\tau), n_0(\tau))$ by

$$B_0(\tau) = \omega_{\text{body}}/(\omega_{\text{space}} \cdot e_z) \equiv (X^{-1}\dot{X})^\vee/(\dot{X}X^{-1})^\vee \cdot e_z, \qquad (4.7)$$

$$n_0(\tau) = X^{-1}(\tau)\, e_z, \qquad (4.8)$$

and define the moved ribbon by

$$B_t(\tau) = X(t)\, B_0(\tau), \qquad n_t(\tau) = X(t)\, n_0(\tau).$$

The ribbon rolls without sliding. Indeed,

1) The base point lies on the plane $z = 1$: Using (4.6) we observe that

$$X(X^{-1}\dot{X})^\vee = (\dot{X}X^{-1})^\vee, \quad \text{that is,} \quad X\omega_{\text{body}} = \omega_{\text{space}}, \qquad (4.9)$$

so that $B_t(t) = \omega_s/\omega_s \cdot e_z = (\omega_x/\omega_z, \omega_y/\omega_z, 1)$, as asserted.

2) $n_t(t) \perp \{z = 1\}$: indeed, $n_t(t) = e_z$.

3) $B_t(\tau)$ is tangent to $\{z = 1\}$:

$$\frac{d}{d\tau} B_t(\tau)|_{\tau=t} = X(t)\frac{d}{d\tau} B_0(\tau) = X(t)\frac{d}{d\tau}(\omega_b/\omega_s \cdot e_z)$$

$$\overset{(4.9)}{=} X(t)\frac{d}{d\tau}[X^{-1}\omega_s(\tau)/(\omega_s \cdot e_z)]$$

$$= X(t)[-(X^{-1}\dot{X}X^{-1})\,\omega_s]/\omega_s \cdot e_z + X(t)\,X^{-1}(\tau)\frac{d}{d\tau}[\omega_s/\omega_s \cdot e_z].$$

We set now $t = \tau$; the first term in the last sum vanishes because of the identity

$$A\check{A} = 0, \qquad (4.10)$$

while in the last term we are differentiating a vector whose z-component is constant. This proves that the base curve is tangent to the plane.

4) There is no sliding:

$$\frac{d}{dt} B_t(\tau)|_{\tau=t} = \dot{X}X^{-1}\omega_s/\omega_s \cdot e_z \equiv \hat{\omega}_s\omega_s/\omega_s \cdot e_z \overset{(4.10)}{=} 0.$$

5) It remains to prove that with the extra periodicity conditions, the ribbon is closed. The only nontrivial part of this statement is to show the periodicity of the denominator $\omega_s \cdot e_z$ in the definition of $B_0(\tau)$. Using the identity (4.9), we obtain

$$\omega_s(T)\cdot e_z \overset{(4.9)}{=} X(T)\,\omega_b(T)\cdot e_z = X(T)\,\omega_b(0)\cdot e_z = \omega_b(0)\cdot X^{-1}(T)\,e_z = \omega_s(0)\cdot e_z,$$

where we used the assumption $X(T)\,e_z = e_z$ together with $\omega_s(0) = X(0)\,\omega_b(0) = \omega_b(0)$ in the last step.

The proof of $k_r = k_p - \omega_z/v$ is identical to the proof of Theorem 2, and we omit it. In fact, the only data about the surface used in that Theorem are the curve C_S and the normal vectors to the surface along C_S (i.e., the Gauss image of C_S).

Finally, the last formula of the theorem could be proved by interpolating the ribbon by a disk in $\mathbb{R}^3$ (so that the $n_0(\tau)$ are normal to the disk at its boundary $B_0(\tau)$), and applying the Gauss-Bonnet Theorem according to which $\iint_{\text{disk}} K\, d\sigma + \int_{B_0} k_r\, ds = 2\pi$, where K is the Gaussian curvature of the disk, which by the definition of that curvature is the area on the unit sphere lying to the left of the curve traced out by the unit normal vectors, with "left" defined with respect to the same normal n that is used in the definition of the geodesic curvature of B_0. Strictly speaking, one has to prove that the ribbon can be interpolated by a disk, but the disk is not really necessary for either the statement or the application of the Gauss-Bonnet Theorem – all that is needed is the ribbon. This is explained in detail in [L]. □

Proof of Corollaries 3 and 4. Applying Theorem 1 to the Poinsot description and noting that the angular momentum m points in the direction opposite to that of the z-axis we obtain

$$\phi = -\alpha \overset{(1.1a)}{=} -\int_0^L k(\Gamma_E)\, ds - \omega_z T (\text{mod } 2\pi) = \Omega + \omega_m T (\text{mod } 2\pi), \quad (4.11)$$

where Ω is the area of the image of $\text{Int}(\Gamma_E)$ under the Gauss map, where the interior is defined by the inward unit normal, as in Theorem 1. We recall that the Gauss image of Γ_E is given by $\mu^{-1}M(t)$, the normalized angular momentum vector. Observing the direction of the trajectories on the ellipsoid E and on the momentum sphere (Figs. 2 and 3) we conclude that

$$\Omega = \begin{cases} A & \text{if } E_1 > E > E_2, \\ 4\pi - A & \text{if } E_2 > E > E_3, \end{cases} \quad (4.12)$$

It remains to prove (2.2). The Gauss-Bonnet formula [D] gives $\Omega = 2\pi - \int k(\Gamma_e)$, so that

$$A = \begin{cases} \Omega = 2\pi - \int k(\Gamma_E) & \text{if } E > E_2, \\ 4\pi - \Omega = 2\pi + \int k(\Gamma_E) & \text{if } E < E_2. \end{cases}$$

Since $0 < A < 2\pi$, we conclude that $\int k\, ds > 0$ in the first case and $\int k\, ds < 0$ in the second, so that $A = 2\pi - \left|\int k(\Gamma_E)\, ds\right|$. □

5. Appendix: The Dynamics of a Free Rigid Body

To keep the exposition self-contained, we sketch the Poinsot characterization of the free rigid body motion; the details can be found in most mechanics books, for instance, in [A] or [G]. We recall that with any rigid body one can associate an ellipsoid centered at the body's center of mass and rigidly affixed to the body, called the *ellipsoid of inertia* [A]. We choose an inertial frame in which the center of mass of the body is at rest and at the origin. In terms of the ellipsoid of inertia the motion of the body with no external forces can be expressed particularly elegantly, according to POINSOT [A]:

In a free motion of a rigid body the ellipsoid of inertia rolls without sliding along a stationary plane P, called the invariable plane [G]. *In particular, the distance from the center of the ellipsoid to the plane P is constant. The plane P is perpendicular to the vector of angular momentum.* Let the vector connecting the origin with the point of contact be denoted by ρ. Then the angular velocity ω is given by $\omega = \sqrt{2E}\,\rho$, where E is the kinetic energy of the motion.

This description has a simple consequence: $\omega_m = \text{const}$, where ω_m is the component of ω normal to the invariable plane. Indeed, by the above description, $\omega_m = \sqrt{2E}\,\rho_3$ is constant since the last component of ρ, i.e., the distance from the center of the ellipsoid to the invariable plane, does not change. Considering now some particular motion of the body, we look at the point of the ellipsoid E which is in contact with the plane P at $t = 0$. As the ellipsoid rolls along P (while its center remains stationary), the chosen point will touch P again at some time $T > 0$; the result of this relatively complicated motion

is simple: it is just a rotation of the rigid body around the line of the angular momentum vector m through some angle ϕ. Let Γ_E and Γ_P be the two curves, one on the ellipsoid of inertia, the other on the plane, both traced out by the point of contact. These two curves are parametrized naturally by the time t. The curve Γ_E on the ellipsoid is called the *polhode*, while the planar curve Γ_P is called the *herpolhode*. Let $k_E(t)$ denote the geodesic curvature* of Γ_E and let $k_P(t)$ be the curvature of Γ_P. These play a key role in the discussion in the section above.

An even simpler (although not as complete a) description of the motion of a free rigid body can be achieved by the process of reduction, i.e., by "ignoring" its position relative to the space frame but rather affixing a coordinate frame to the body itself and keeping track of the motion of the angular momentum vector with respect to the new non-inertial frame. Let $M = (M_1, M_2, M_2)$ be the angular momentum vector expressed in the body frame, and let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the angular velocity expressed in that frame. These vectors enjoy the key relationship $M = I\Omega$, where I is a constant positive definite symmetric matrix called the tensor of inertia of the body. The energy $E = \frac{1}{2}(I^{-1}M, M)$ and the length $|M| = \mu$ are the two constants of motion for the reduced system. Geometrically, this shows that the vector $M(t)$ moves along a line of intersection of the sphere $|M| = \mu$ and the energy ellipsoid $(I^{-1}M, M) = 2E$; this motion is shown in Fig. 8. For simplicity we choose the axes of the body frame to coincide with the principal axes of inertia, thus diagonalizing the tensor of inertia: $I = \text{diag}(I_1, I_2, I_3)$, and we assume without loss of generality that the principal moments of inertia satisfy $I_1 < I_2 < I_3$.

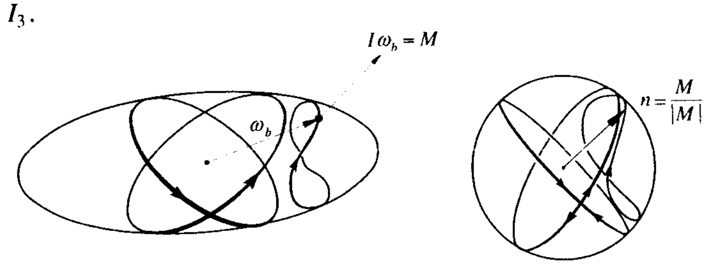

Fig. 8. The dynamics of the rigid body on the momentum sphere. Trajectories of $n = M/|M|$ on the unit sphere are Gauss' images of the polhodes.

All the trajectories $M(t)$ on a fixed momentum sphere $|M| = \mu$ with the exception of the saddle-saddle connections are closed. These exceptional heteroclinic solutions have the energy $E = \mu^2/2I_2$. Fig. 2 shows that the nutation period T from the Poinsot description coincides with the period of $M(t)$, the body angular momentum. Indeed, the position of the angular momen-

* We recall that the geodesic curvature of a curve on a surface at each point can be defined as the curvature of the orthogonal projection of the curve onto the tangent plane to the surface at the point in question. The geodesic curvature thus measures the curvature of a curve "within" the surface.

tum vector relative to the body repeats itself exactly after one "period" T of the polhode. To relate this geometrical picture to the Gauss map, we note that $M = \text{grad}\,\frac{1}{2}(I\Omega, \Omega)$, so that $n = M/|M|$ is the unit vector normal to the ellipsoid of inertia $(I\Omega, \Omega) = 2E$. Putting it differently,

The Gauss map takes the polhode Γ_E onto the curve on the unit sphere traced out by the normalized vector $n = M/|M|$ of angular momentum.

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References

[A] V. I. ARNOL'D, *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1988.

[BH] M. V. BERRY & J. H. HANNAY, Classical non-adiabatic angles. *J. Phys. A: Math. Gen.* **21** (1988) L325–331.

[B] M. V. BERRY, Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. A* **392** (1984), 45–57.

[BKMS] A. BLOCH, P. S. KRISHNAPRASAD, J. MARSDEN & G. SANCHEZ DE ALVAREZ, Stabilization of rigid body dynamics by internal and external torques. *Automatica* (to appear).

[C] R. Y. CHIAO, Optical manifestations of Berry's topological phases: Aharonov-Bohmlike effect for the photon. *Proc. 3rd International Symposium on Foundations of Quantum Mechanics*, to be published by the Physical Society of Japan. May 1989.

[D] M. P. DOCARMO, *Differential Geometry of curves and surfaces.* Prentice Hall, 1976.

[G] H. GOLDSTEIN, *Classical Mechanics.* Addison-Wesley, 1950.

[GKM] S. GOLIN, A. KNAUF & S. MARMI, The Hannay angles: geometry, adiabaticity, and an example, *Comm. Math. Phys.* **123** (1989), 95–122.

[H] J. H. HANNAY, Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian. *J. Phys. A: Math. Gen.* **18** (1985), 221–230.

[KS] M. KUGLER & S. SHTRIKMAN, Berry's phase, locally inertial frames, and classical analogues. *Phys. Rev. D.* **37** (1988), 934–937.

[L] M. LEVI, A "bicycle wheel" proof of the Gauss-Bonnet theorem, dual cones and some mechanical manifestations of the Berry phase, to appear in *Expos. Math.*

[MMR] J. MARSDEN, R. MONTGOMERY & T. RATIU, Reduction, Symmetry and Berry's phase in Mechanics, *Memoirs of the Amer. Math. Soc.*, **436** (1990), 1–110.

[M] R. MONTGOMERY, The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its Generalization to the non-integrable case. *Comm. Math. Phys.*, **120** (1988), 269–294.

[M1] R. MONTGOMERY, By how much does the rigid body rotate?, A Berry's phase from the 18th Century. *Am. J. Phys.* **59** (1991), 394–398.

[S] B. SIMON, Holonomy, the quantum adiabatic theorem and Berry's phase. *Phys. Rev. Lett.* **5(24)** (1983), 2167–2170.

[W] E. T. WHITTAKER, *A Treatise on Analytical Mechanics*, Cambridge University Press, 1917.

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