Slings and Spirals

Spirals whose curvatures are exponential functions of the arclength are denser near their centers than logarithmic spirals (see Figure 1). I stumbled upon these exponential spirals thanks to an inspiring conversation with Tadashi Tokieda.

In a previous article, I discussed the following parody of the sling [1]. If we want to spin a stone held by a rope in a circle with a hand that is spinning in a smaller concentric circle—where the rope forms a fixed angle $\alpha > \pi/2$ with the hand's velocity—then we must accelerate the hand in direct proportion to its speed: $\dot{v} = cv^2$ (with some $c = \text{const.}$ that depends on the circles' radii and on α). Such $v = v(t)$ blows up in finite time, meaning that maintaining constant α is possible only for that long.

Here is a reasonable modification of the problem: *What path must my hand take if I no longer insist on moving it in a circle, but instead move it with constant speed (say, $v = 1$) while still keeping $\alpha = \text{const.}$?*[1] An exponential spiral turns out to be the answer, and the following is a precise statement:

MATHEMATICAL CURIOSITIES

By Mark Levi

Tracking at a Constant Angle

Consider the pendulum (i.e., a point mass on a massless rod) in the plane, with no gravity (see Figure 2). If the pivot P moves with constant speed $v = 1$ in such a way as to keep the angle α in Figure 2 constant, then the path of P is an exponential spiral with curvature

$$k = \omega_0 e^{-as}, \quad \text{where} \quad a = \frac{\cos\alpha}{r}. \qquad (1)$$

Here, $s = t$ is the arclength = time (recall that $v = 1$), r is the rod length, and ω_0 is the initial angular velocity of the pendulum.

I omit the proof (available upon request).

[1] As before, α is the angle between the hand's velocity vector and the rope.

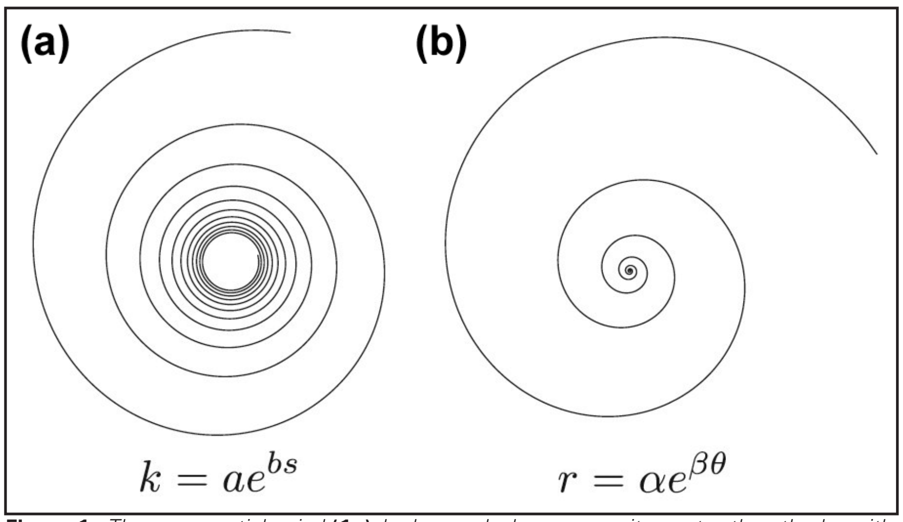

Figure 1. *The exponential spiral* **(1a)** *looks much denser near its center than the logarithmic spiral* **(1b)**. *In particular, the length around the center of the former spiral lying inside any disk is infinite, in contrast to the logarithmic spiral.*

Some Observations

1. If α is obtuse (as in Figure 2) so that the pendulum lags, then $k(s) \to \infty$ as $s \to \infty$ according to (1). The pivot moves in a tightening spiral to keep $\alpha = \text{const.}$

2. The curvature k plays a double role in that it is also the angular velocity of the rod. Indeed, the angle between the rod and the velocity $\mathbf{v}_P$ is constant. For obtuse α, k grows exponentially and so does the angular velocity of the rod. The bob then spirals towards a circle of radius r, while P spirals towards a point. And since the bob's speed approaches infinity, so does the force with which the hand P pulls on the rod.

3. The hand that holds P expends power (work per unit of time) given by $-T\cos\alpha$, where T is the tension of the rod. Note that *the hand does work thanks to the fact that the velocity of P has nonzero projection onto the direction of the force, i.e., the direction of the rod*. When swinging an axe overhead while splitting wood, we give the axe its kinetic energy by exactly the same mechanism.[2] This is also how one can throw a stick with great speed.

[2] Except, perhaps, for the initial part of the swing when one applies torque to the handle to begin the swing.

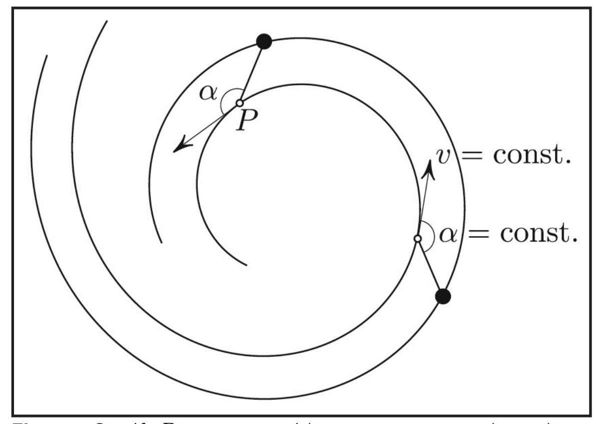

Figure 2. *If P moves with constant speed to keep $\alpha = \text{const.}$, then the curvature of P's path is an exponential function of time (and thus of arc length).*

Some Questions

1. At what rate do the coils of an exponential spiral accumulate? (Providing a precise meaning of this rate is part of the question.)

2. Does an exponential spiral have an asymptote? More generally, consider a curve that is defined by its curvature $k = k(s)$. How fast must k approach 0 as $s \to \infty$ (or $-\infty$) for the curve to have an asymptote?

Do exponential spirals arise in any other settings? It would seem likely that such natural objects arise somewhere in nature, but I couldn't find any other examples.

References

[1] Levi, Mark. (2018, January 29). Slings, bullets, blow-up, and linearity. *SIAM News*, *51*(1), p. 6.

The figures in this article were provided by the author.

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