Kelvin's Circulation Theorem and an Isoperimetric Inequality

MATHEMATICAL CURIOSITIES
By Mark Levi

Kelvin's theorem in two dimensions states that the vorticity (the curl of the velocity field) in an inviscid incompressible fluid is transported by the fluid—that is, it remains constant along the trajectories of the fluid particles. The following intuitive explanation, I think, shows what's really going on.

We begin by dyeing a small circular disk of fluid at $t = 0$, as shown in Figure 1. The surrounding fluid applies torque $\tau = 0$ to our disk (relative to the disk's center); this is so because the tangential component of the force on the boundary is zero for the inviscid fluid, by the definition of viscosity. For the angular momentum L of the blob (computed relative to the blob's center), we have:

$$dL/dt = \tau = 0 \text{ at } t = 0. \qquad (1)$$

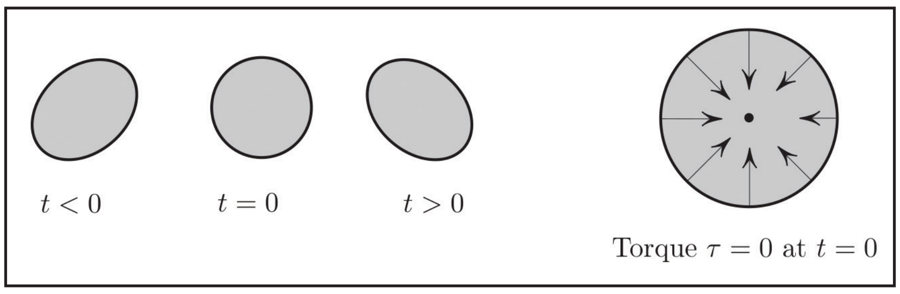

Figure 1. *Kelvin's theorem by (instantaneous) conservation of angular momentum.*

The vorticity $\omega = 2\Omega$ is twice the average angular velocity Ω of the radii of the disk; (1) thus becomes

$$d/dt\,(I\Omega) = \dot{I}\,\Omega + I\,\dot{\Omega} = 0 \text{ for } t = 0,$$

where I is the moment of inertia of the blob. But $\dot{I}(0) = 0$, as the circular disk minimizes the moment of inertia among all other shapes of the same area (here we use the incompressibility). We conclude that $\dot{\Omega} = 0$ and thus $\dot{\omega} = 0$.

In summary, Kelvin's 2D theorem boils down to two facts:

- zero torque from the surrounding fluid on a round disk, and
- the isoperimetric inequality:

$$\int_{x^2+y^2\le r^2} (x^2 + y^2)\,dxdy \le \int_D (x^2 + y^2)\,dxdy$$

valid for all domains D of area πr^2. Standard analytical proofs rely on vector identities and tend to obscure this intuition.

An interesting and non-trivial exercise is to extend this approach to the proof of the 3D version of Kelvin's theorem (according to which vorticity stretches with the fluid, i.e., satisfies the linearized equation).

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University. *The work from which these columns are drawn is funded by NSF grant DMS-1412542.*