Physica D 244 (2013) 68–73

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Traveling waves in chains of pendula

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ARTICLE INFO

Article history:
Received 11 June 2012
Received in revised form
29 October 2012
Accepted 30 October 2012
Available online 6 November 2012
Communicated by J. Bronski

Keywords:
Coupled pendula
Traveling wave
Saddle–node bifurcation
Frenkel–Kontorova model

ABSTRACT

The existence of traveling wave solutions for the discrete, forced, damped sine-Gordon equation, which serves as a model of arrays of Josephson junctions and coupled pendula, in the case of small coupling coefficient has been addressed before. In this paper we prove the existence of a discrete traveling wave in a lattice of coupled pendula with a large coupling coefficient in the presence of damping and forcing, and show the global stability of this wave.

1. Introduction and a heuristic discussion

The forced damped sine-Gordon equation

$$\varphi_{tt} + c\varphi_t - \varphi_{ss} + \sin\varphi = I, \tag{1.1}$$

where c and I are constants, arises in many physical applications such as nonlinear resonant optics [1,2] and Josephson junctions [3]. Space discretization of this equation, obtained by replacing φ_{ss} by the second difference

$$\ddot{x}_i + c\dot{x}_i + \sin x_i = k(x_{i+1} - 2x_i + x_{i-1}) + I, \quad i \in \mathbb{Z} \tag{1.2}$$

is of special interest from a physical point of view: it serves as a model of infinite arrays of coupled pendula [4], arrays of Josephson junctions [5], or as a dynamical Frenkel–Kontorova model of electrons in a crystal lattice [6]. Eq. (1.2) describes the motion of an array of pendula each of which is coupled to its nearest neighbors by a torsional spring with a coupling coefficient k. In addition, each pendulum is subject to a constant torque I and to a viscous drag with a drag coefficient c. The angles x_i formed by the ith pendulum with the vertical axis evolve according to (1.2), assuming physical units have been scaled appropriately.

One of the interesting and important features of the sine-Gordon PDE is the existence of traveling wave solutions [7], or "kinks", which have been extensively studied both analytically and numerically [8–12]. Proving the existence of such kinks reduces to finding a heteroclinic solution for an ODE. By contrast, proving the existence of traveling waves in a discretization such as (1.2) is a more delicate problem since the translation group under which the system is invariant is discrete. The traveling wave in a lattice is, by definition, the solution $\mathbf{x}$ of (1.2) satisfying

$$x_i(t+T) = x_{i+1}(t) \quad \text{for all } t \in \mathbb{R}$$

for some $T > 0$. If we denote $x_0(t) = \varphi(t)$, then the above definition is equivalent to $x_i(t) = \varphi(t - iT)$ (see [13,14]).

The existence of discrete traveling waves has been proven for small k and a specific range of I, see [4]; in these traveling waves only one pendulum rotates at a time, one after the other, and the wave thus travels down the chain. This wave turns out to be orbitally stable. As an interesting aside, the spatially discrete model exhibits a feature not present in the continuous mode, that of the coexistence of two stable traveling waves of different speeds [4]. Despite its interesting behavior, the case of small k provides no information about the PDE since in the discretization $\varphi_{ss} \approx \frac{1}{h^2}(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1})$ the coefficient $k = \frac{1}{h^2}$ is large. For large k one cannot expect each pendulum to be "too independent" of its neighbors. Rather, one expects traveling waves with many pendula participating in the motion at any given time (see Fig. 1), as opposed to the case of $k \ll 1$ where only one pendulum rotates at a time. Watanabe et al. [15] studied the existence of traveling waves for large values of k both numerically and experimentally. In this paper, we prove such an existence for large k and sufficiently small gravitational acceleration. Among other results on traveling waves in lattice differential equations [16–22], there is a work by Katriel [23] about the existence of traveling waves in Eq. (1.1), where it was shown, in particular, that for any c and k a traveling wave of any

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doi:10.1016/j.physd.2012.10.007

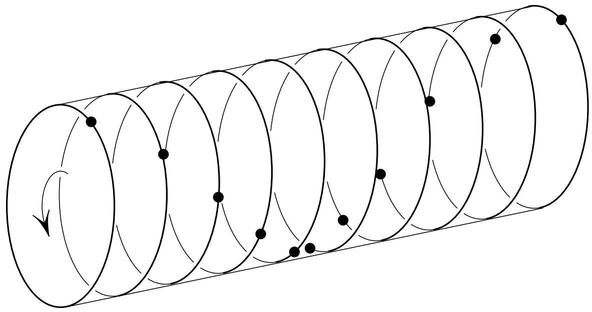

Fig. 1. A traveling wave in a chain of pendula with nearest-neighbor torsional coupling, in the presence of forcing and damping.

velocity (i.e., with any T) can be realized by an appropriate choice of I. The proof is based on reducing the existence of a traveling wave to a difference–differential equation for $\varphi(t) = x_0(t)$ and applying a fixed point theorem from functional analysis. This approach does not, however, give information on the dynamics of other solutions besides the traveling wave.

In this paper we consider the system

$$\ddot{x}_i + c\dot{x}_i + \varepsilon \sin x_i = k(x_{i+1} - 2x_i + x_{i-1}) + I; \quad i \in \mathbb{Z} \tag{1.3}$$

which differs from (1.2) in that the gravitational acceleration is ε rather than 1. We also impose the periodic boundary condition

$$x_{i+N} = x_i + 2\pi; \quad i \in \mathbb{Z} \tag{1.4}$$

which corresponds to a discrete kink; here N is an arbitrary positive integer. Instead of an infinite chain, we can thus think of a ring of N coupled pendula with a twist of 2π (see Fig. 1).

The system of Eqs. (1.3) can then be rewritten in the vector form

$$\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + \nabla V(\mathbf{x}) = \mathbf{0}; \quad \mathbf{x} = (x_1, x_2, \ldots, x_N), \tag{1.5}$$

where the potential V is given by

$$V(\mathbf{x}) = \sum_{i=1}^{N} \left[\frac{1}{2} k(x_{i+1} - x_i)^2 - \varepsilon \cos x_i - I x_i \right], \quad x_{N+1} = x_1 + 2\pi. \tag{1.6}$$

Thus the system of pendula in Fig. 1 can be thought of as a particle $\mathbf{x} \in \mathbb{R}^N$ moving in the potential V sketched in Fig. 2. Note that the torque I controls the "slant" of the potential, and one expects that for I above a certain critical value the particle will have to slide down the trough of the potential. This motion is born of a saddle–node bifurcation as suggested by Fig. 2. This motion corresponds to the traveling wave in the chain of pendula.

The bifurcating equilibria are illustrated more specifically in Fig. 3. For I slightly above a critical value, the traveling wave behaves as follows: the pendula appear to be almost at rest for most of the time; during transitional intervals each pendulum moves clockwise (in this figure), and takes up the previous position of its neighbor. When observing numerical simulations, the system appears to be almost at rest most of the time and to undergo relatively sudden transitions. Such behavior is characteristic of the trajectories passing the vicinity of the "shadow" of an equilibrium destroyed in a saddle–node bifurcation. As mentioned, this "jerky" behavior occurs for I slightly above the critical value. Our results, however, do not put any upper bounds on the torque I.

Remark 1. Heuristic arguments and computational evidence suggest that the sinusoidal potential in our discretized sine-Gordon equation has an interesting special feature: the critical saddle–node value of the torque required to dislodge the pendula from equilibrium is exponentially small in N (the number of pendula): $I_N = O(\varepsilon^N)$. This phenomenon is reminiscent of the sharpness of resonance zones in the Mathieu equation [24–26]. In this paper, we show that the estimate holds for $N = 3$, planning to address the general case in a future work.

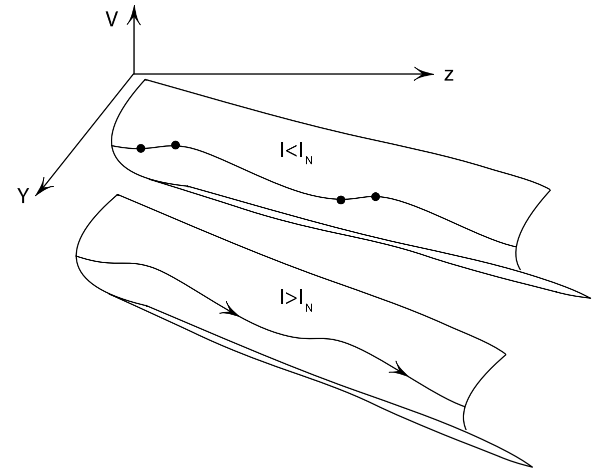

Fig. 2. Traveling wave viewed as the motion of a particle in a potential field.

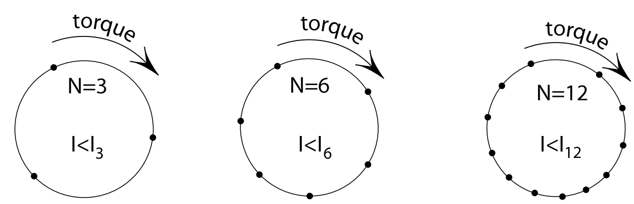

Fig. 3. The chain of pendula sags under the gravitational potential. When the torque overcomes this sag, a traveling wave is born.

Remark 2. The system of Eqs. (1.3) can be interpreted as a dynamical version of the Frenkel–Kontorova model [6], and the saddle–node bifurcations giving birth to traveling waves can be interpreted as de-pinning of electron densities.

2. Results

The periodicity condition (1.4) reduces the system with infinitely many pendula to one with N degrees of freedom:

$$\begin{cases} \ddot{x}_1 + c\dot{x}_1 + \varepsilon \sin x_1 = x_2 + x_N - 2x_1 + I - 2\pi \\ \ddot{x}_2 + c\dot{x}_2 + \varepsilon \sin x_2 = x_3 + x_1 - 2x_2 + I \\ \vdots \\ \ddot{x}_N + c\dot{x}_N + \varepsilon \sin x_N = x_1 + x_{N-1} - 2x_N + I + 2\pi. \end{cases} \tag{2.1}$$

Here we take $k = 1$ for simplicity; the treatment remains the same for any fixed k.

Our system (2.1) is invariant under simultaneous translation τ of all the angles by 2π:

$$\tau\mathbf{x} = \tau(x_1, \ldots, x_N) = (x_1 + 2\pi, \ldots, x_N + 2\pi) = \mathbf{x} + 2\pi\mathbf{1}, \quad \mathbf{1} = (1, \ldots, 1).$$

Theorem 2.1. *There exists $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$ and for every integer $N \geq 2$, system (2.1) has a globally attracting invariant circle K in the phase space $\mathbb{R}^N (\text{mod } \tau) \times \mathbb{R}^N$. This circle admits a parametric representation $(\mathbf{x}(z), \mathbf{v}(z)) \in \mathbb{R}^{2N}$ with $\mathbf{x}(z + 2\pi) = \mathbf{x} + 2\pi\mathbf{1}$, $\mathbf{v}(z + 2\pi) = \mathbf{v}(z)$ (see Fig. 4).*

According to this theorem, all the equilibria of system (2.1), if any, lie on the invariant circle. This essentially gives a complete description of the system, leaving only two options: (i) if the torque I is too large for equilibria to exist, then there is a globally orbitally stable periodic solution (traveling wave), or (ii) if there exist equilibria, they must lie on the invariant circle, and every solution approaches an equilibrium.

Theorem 2.2. *There exists $\epsilon_1 > 0$ such that for every $0 < \epsilon < \epsilon_1$ the following holds. There exists a constant $I_N = I_N(\varepsilon)$ such that for each $I > I_N(\epsilon)$ system (2.1) has a unique traveling wave solution, i.e., a solution $\mathbf{x}(t)$ and $T > 0$ such that for all t we have*

$$x_i(t+T) = x_{i+1}(t), \tag{2.2}$$

where $x_i(t)$ is defined for all $i \in \mathbb{Z}$ via $x_{i+N} = x_i + 2\pi$.

Eq. (2.2) implies that each pendulum takes up the position of its next neighbor after time T. Applying this equation N times, we conclude that $x_i(t+NT) = x_{i+N}(t) = x_i(t) + 2\pi$, i.e., after time NT the chain "tumbles over" once.

Computational evidence suggests that $I_N(\epsilon) = C_N\varepsilon^N + O(\varepsilon^{N+1})$, where $C_N > 0$.

Theorem 2.3. *Consider the case of $N = 3$, and let $0 < \epsilon < \epsilon_1$, where ϵ_1 is from Theorem 2.2. There exists a function $I_3(\varepsilon) = \frac{\epsilon^3}{72} + O(\epsilon^4)$ such that system (2.1) undergoes a saddle–node bifurcation at $I_3 = I_3(\varepsilon)$, i.e., for $I > I_3(\varepsilon)$ the system possesses a globally attracting traveling wave solution and for $I < I_3(\varepsilon)$ every solution approaches an equilibrium.*

3. Proofs

Proof of Theorem 2.1. The heuristic idea of the proof is illustrated in Fig. 4. The proof exploits the fact that the flow of system (2.1) in $\mathbb{R}^{2N}$ contracts in a direction transversal to the direction of simultaneous rotation of all the pendula, in some precise sense. This contraction is constantly perturbed by the nonlinear terms in (2.1), but if ε is sufficiently small, this perturbation is dominated by the contraction. We now make all this precise.

New variables. The following notations are used in the proofs of the above theorems. Let $z_i = x_i - \frac{2\pi}{N}(i-1)$; note that z_i measures the deflection from the linear distribution of angles. Set $z = \frac{1}{N}\sum_{i=1}^{N} z_i$ and define

$$y_i = z_i - z. \tag{3.1}$$

Note that $\sum_{i=1}^{N} y_i = 0$. With the new variables (3.1), the system of Eqs. (2.1) can be written in the vector form

$$\ddot{z}\mathbf{1} + \ddot{\mathbf{y}} + c\dot{z}\mathbf{1} + c\dot{\mathbf{y}} + \varepsilon\sin(\mathbf{y} + z\mathbf{1} + \mathbf{v_0}) - \Delta\mathbf{y} = I\mathbf{1} \tag{3.2}$$

where $\mathbf{1} = (1,1,\ldots,1)^t$, $\mathbf{y} = (y_1, y_2, \ldots, y_N)^t$, $\mathbf{v_0} = (0, \frac{2\pi}{N}, \ldots, \frac{2\pi}{N}(N-1))^t$, and Δ is the discrete Laplacian: $(\Delta\mathbf{y})_i = y_{i+1} - 2y_i + y_{i-1}$.

To establish the existence of a stable traveling wave solution (for some values of ε and I) it suffices to show that system (3.2) has an attracting invariant orbit $(z(t), \mathbf{y}(t))$, as illustrated in Fig. 4, such that for some $T > 0$ we have

$$z(t+T) = z(t) + 2\pi, \qquad \mathbf{y}(t+T) = \mathbf{y}(t) \quad \text{for all } t.$$

Let $Y := \mathbf{1}^{\perp}$. Note that $\mathbf{1} \in \ker(\Delta)$, $Y = \operatorname{ran}(\Delta)$, and $\Delta_Y : Y \to Y$ (the restriction of Δ to the invariant subspace Y) is an invertible linear map. By projecting Eq. (3.2) first on $\mathbf{1}$ and then on Y, we obtain the equivalent system for the "center of mass" z and the "shape" $\mathbf{y}$:

$$\ddot{z} + c\dot{z} + \frac{\varepsilon}{N}\sum_{i=1}^{N}\sin(y_i + z + v_{0i}) = I, \tag{3.3}$$

$$\begin{aligned}&\ddot{\mathbf{y}} + c\dot{\mathbf{y}} + \varepsilon\pi_Y(\sin(\mathbf{y} + z\mathbf{1} + \mathbf{v_0})) - \Delta_Y\mathbf{y} = 0;\\ &\sum_{i=1}^{N} y_i = 0,\end{aligned} \tag{3.4}$$

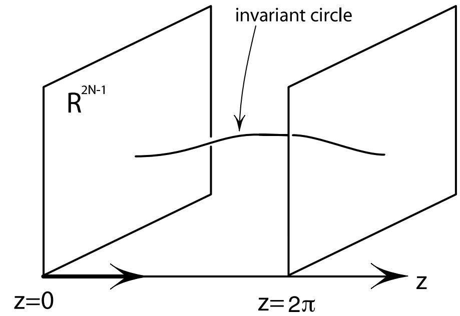

Fig. 4. Schematic representation of an invariant circle in $\mathbb{R}^{2N}$.

where $\pi_Y : \mathbb{R}^N \to Y$ is the orthogonal projection. Eqs. (3.3) and (3.4) are then equivalent to the first order system

$$\begin{cases}\dot{z} = u\\ \dot{u} = -cu - \dfrac{\varepsilon}{N}\displaystyle\sum_{i=1}^{N}\sin(z + y_i + v_{0i}) + I\\ \dot{\mathbf{y}} = \mathbf{v}\\ \dot{\mathbf{v}} = -c\mathbf{v} - \varepsilon\pi_Y(\sin(\mathbf{y} + z\mathbf{1} + \mathbf{v_0})) + \Delta_Y\mathbf{y}.\end{cases} \tag{3.5}$$

Introducing new variables z_1, u_1 via $z_1 := z + \frac{1}{c}u - \frac{I}{c^2}$, $u_1 := u - \frac{I}{c}$, we recast (3.5) into the form

$$\dot{\mathbf{w}} = L\mathbf{w} + \boldsymbol{\omega} + \varepsilon\boldsymbol{R}(\mathbf{w}), \tag{3.6}$$

where $\mathbf{w} = (z_1, u_1, \mathbf{y}, \mathbf{v})^t$, $\boldsymbol{\omega} = (\frac{I}{c}, 0, \ldots, 0)^t$, $\boldsymbol{R}$ is the vector of non-linear terms in (3.5) after applying the change of variables z_1 and u_1, and

$$L = \left[\begin{array}{c|c} 0 & 0\\ \hline 0 & \Lambda\end{array}\right], \qquad \Lambda = \left[\begin{array}{c|cc} -c & 0 & \\ \hline 0 & 0 & I\\ & \Delta_Y & -cI\end{array}\right],$$

with I being the $(N-1)\times(N-1)$ identity matrix. The form of (3.6) suggests that the flow is a combination of translation in the direction of z_1 and a contraction in the remaining directions. We make this more precise by the following lemma.

Lemma 3.1. *The time-1 map of system (3.6) is given by:*

$$\varphi^1 : \begin{cases} z_1 \mapsto z_1 + \dfrac{I}{c} + \varepsilon\alpha(z_1, \mathbf{u}, \varepsilon)\\ \mathbf{u} \mapsto e^{\Lambda}\mathbf{u} + \varepsilon\beta(z_1, \mathbf{u}, \varepsilon),\end{cases} \tag{3.7}$$

where $\mathbf{u} = (u_1, \mathbf{y}, \mathbf{v})$. The functions α and β defined on $\{(z_1, \mathbf{u}, \varepsilon);\ z_1 \in \mathbb{R},\ \|\mathbf{u}\| < 1,\ 0 < \varepsilon < 1\}$ are bounded in C^1-norm.

Proof of Lemma 3.1. Applying the variation of constants formula to the solution of (3.6) with the initial condition $\mathbf{w}$, we obtain

$$\underbrace{\varphi^1\mathbf{w}}_{(z_1,\mathbf{u})^t} = e^L\mathbf{w} + \boldsymbol{\omega} + \varepsilon\underbrace{\int_0^1 e^{(1-\tau)L}\boldsymbol{R}(\varphi^\tau\mathbf{w})\,d\tau}_{(\alpha,\beta)^t}, \tag{3.8}$$

where the remainder is bounded by

$$\left|\int_0^1 e^{(1-\tau)L}\boldsymbol{R}(\varphi^\tau\mathbf{w})\,d\tau\right| \le \max_{0\le\tau\le1}|\boldsymbol{R}(\varphi^\tau\mathbf{w})| \le c_1, \tag{3.9}$$

for some constant $c_1 > 0$. Thus α and β are bounded in sup norm. Now we show that their derivatives are also bounded. Indeed, the Jacobian matrix $\Phi(t, \mathbf{w}) := \frac{\partial}{\partial\mathbf{w}}\varphi^t\mathbf{w}$ satisfies

$$\begin{aligned}\dot{\Phi} &= \frac{\partial}{\partial t}\frac{\partial}{\partial\mathbf{w}}\varphi^t\mathbf{w} = \frac{\partial}{\partial\mathbf{w}}\frac{\partial}{\partial t}\varphi^t\mathbf{w}\\ &= \frac{\partial}{\partial\mathbf{w}}\left(L\varphi^t\mathbf{w} + \boldsymbol{\omega} + \varepsilon\boldsymbol{R}(\varphi^t\mathbf{w})\right) = \left(L + \varepsilon\boldsymbol{R}'(\boldsymbol{\varphi^t}\mathbf{w})\right)\Phi,\end{aligned}$$

where $\boldsymbol{R}'$ is the derivative of $\boldsymbol{R}$. Thus

$$\frac{\partial}{\partial t}\langle \Phi, \Phi\rangle = 2\langle \dot{\Phi}, \Phi\rangle = 2\langle (L+\varepsilon \boldsymbol{R}'(\boldsymbol{\varphi^t}\mathbf{w}))\Phi, \Phi\rangle \le c_2\langle \Phi, \Phi\rangle$$

for some constant $c_2 > 0$. Since Φ is the Jacobian matrix of φ^t, it follows that α and β have bounded derivatives and thus they are bounded in C^1-norm. □

From here on we omit the subscripts in the variables z_1 and u_1.

Conclusion of the proof of Theorem 2.1. We first show that the map φ^1 given by Eq. (3.7) has an invariant circle, and will then observe that the same circle is preserved by φ^t. We seek this circle as the graph of $f : \mathbb{R} \to \mathbb{R}^{2N-1}$ with $f(z+2\pi) = f(z)$ and invariant under φ^1, i.e. for any $z \in \mathbb{R}$ there exists $\xi \in \mathbb{R}$ such that

$$\phi^1(z, f(z)) = (\xi, f(\xi)), \tag{3.10}$$

or equivalently,

$$f\left(z + \frac{I}{c} + \varepsilon\alpha(z, f(z), \varepsilon)\right) = e^{\Lambda} f(z) + \varepsilon\beta(z, f(z), \varepsilon). \tag{3.11}$$

We note that $\lambda := \|e^{\Lambda}\| < 1$.

For any given Lipschitz function f and a given ϵ, choose $\xi := z + \frac{I}{c} + \varepsilon\alpha(z, f(z), \varepsilon)$ as an independent variable (instead of z). By the implicit function theorem there exists a function g_f such that $z = g_f(\xi, \varepsilon)$. We show that the functional equation (3.11) has a unique solution f in the class of Lipschitz functions. Note that this equation can be rewritten as

$$f(\xi) = e^{\Lambda} f(g_f(\xi, \varepsilon)) + \varepsilon\beta(g_f(\xi, \varepsilon), f(g_f(\xi, \varepsilon)), \varepsilon). \tag{3.12}$$

To prove the existence of f, it suffices to show that the functional operator defined by the right hand side of (3.12):

$$\mathscr{F}(f) := e^{\Lambda} f(g_f) + \varepsilon\beta(g_f, f(g_f)), \tag{3.13}$$

(abbreviating $g_f \equiv g_f(\xi, \varepsilon)$, etc.) acting on the space of Lipschitz functions on $\mathbb{R}$ has a unique fixed point. To this end, we show that $\mathscr{F}$ is a contraction map. We note that $\mathscr{F}$ maps periodic functions of period 2π to periodic functions of the same period. To prove the contraction property, we will estimate

$$\mathscr{F}(f) - \mathscr{F}(h) = e^{\Lambda}\underbrace{\left(f(g_f) - h(g_h)\right)}_{A} + \varepsilon\underbrace{\left(\beta(g_f, f(g_f)) - \beta(g_h, h(g_h))\right)}_{B} \tag{3.14}$$

for any two Lipschitz functions f and h with a Lipschitz constant l. First, we rewrite A as

$$\begin{aligned}|A| &= |f(g_f) - h(g_h)| = |f(g_f) - f(g_h) + f(g_h) - h(g_h)| \\ &\le |f(g_f) - f(g_h)| + |f(g_h) - h(g_h)|.\end{aligned}$$

Since f is a Lipschitz function, from the last inequality we have:

$$|f(g_f) - h(g_h)| \le l|g_f - g_h| + \|f - h\|, \tag{3.15}$$

where $\|\cdot\|$ denotes the sup norm. We need to estimate $|g_f - g_h|$. By the definition of g_f

$$\xi = g_f + \frac{I}{c} + \varepsilon\alpha(g_f, f(g_f)), \qquad \xi = g_h + \frac{I}{c} + \varepsilon\alpha(g_h, h(g_h)).$$

Thus

$$g_f - g_h = \varepsilon\alpha(g_h, h(g_h)) - \varepsilon\alpha(g_f, f(g_f)). \tag{3.16}$$

Now from Lemma 3.1 and using the mean value theorem on the right hand side of (3.16) we have:

$$\begin{aligned}|g_f - g_h| &= \varepsilon|\alpha(g_f, f(g_f)) - \alpha(g_h, h(g_h))| \\ &\le \varepsilon\|\alpha_1\||g_f - g_h| + \varepsilon\|\alpha_2\||f(g_f) - h(g_h)|,\end{aligned}$$

where α_1 and α_2 are the derivatives with respect to the first and second arguments. Thus from the last inequality we get:

$$|g_f - g_h| \le \frac{\varepsilon\|\alpha\|_1}{1 - \varepsilon\|\alpha\|_1}|f(g_f) - h(g_h)|, \tag{3.17}$$

where $\|\alpha\|_1 := \max\{\|\alpha_1\|, \|\alpha_2\|\}$. Now from inequalities (3.15) and (3.17) we have:

$$|A| = |f(g_f) - h(g_h)| \le d\|f - h\|, \tag{3.18}$$

where $d = \frac{1-\varepsilon\|\alpha\|_1}{1-\varepsilon(1+l)\|\alpha\|_1}$. This is the desired estimate of A. Now to estimate B, note that

$$|\beta(g_f, f(g_f)) - \beta(g_h, h(g_h))| \le \|\beta\|_1(|g_f - g_h| + |f(g_f) - h(g_h)|),$$

where $\|\beta\|_1 := \max\{\|\beta_1\|, \|\beta_2\|\}$, and β_1 and β_2 are the derivatives with respect to the first and second arguments. Using estimates (3.17) and (3.18), the last inequality is simplified to:

$$|B| = |\beta(g_f, f(g_f)) - \beta(g_h, h(g_h))| \le d'\|f - h\|, \tag{3.19}$$

where $d' = \frac{\|\beta\|_1}{1-\varepsilon(1+l)\|\alpha\|_1}$. Now using bounds (3.18) and (3.19) in (3.14), we show that $\mathscr{F}$ is indeed a contraction:

$$\begin{aligned}|\mathscr{F}(f) - \mathscr{F}(h)| &= |e^{\Lambda}A + \varepsilon B| \\ &\le d\|e^{\Lambda}\|\|f - h\| + \varepsilon d'\|f - h\| \\ &\le (\lambda d + \varepsilon d')\|f - h\|.\end{aligned}$$

Note that for $\varepsilon < \varepsilon_0 := \frac{1+\lambda(1-2d)}{2d'}$, we have $\lambda d + \varepsilon d' < \frac{1+\lambda}{2} < 1$. Thus $\mathscr{F}$ is a contraction map and has a unique fixed point. This proves the existence of a unique curve K which is invariant under the time-1 map ϕ^1:

$$\phi^1 K = K. \tag{3.20}$$

We claim that K is then invariant under the flow ϕ^t for any time t. Indeed, applying ϕ^t to both sides of (3.20) we get:

$$\phi^t(\phi^1 K) = \phi^t K, \tag{3.21}$$

and since ϕ^t and ϕ^1 commute, we have:

$$\phi^1(\phi^t K) = \phi^t K. \tag{3.22}$$

Now the uniqueness of K implies that:

$$\phi^t K = K, \tag{3.23}$$

as claimed. This concludes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For any $0 < \varepsilon < \varepsilon_0$, where ε_0 is from Theorem 2.1, the system possesses an invariant circle, for any I. We show that for I greater than a certain critical value system (2.1) has no equilibria. This, we claim, implies the existence of a traveling wave solution, i.e., a solution satisfying (2.2). Indeed, since the invariant circle contains no equilibria, it is an orbit of some solution $\mathbf{x}(t)$ (this solution is periodic modulo 2π-translations: $\mathbf{x}(t+p) = \mathbf{x}(t) + 2\pi\mathbf{1}$ for some $p > 0$ and for all t). Since the system is invariant under the translation of index, the translated function $\sigma\mathbf{x}(t) := (x_2, x_3, \ldots, x_N, x_1 + 2\pi)$ is a solution as well, also periodic modulo 2π-translations and thus lying on the invariant circle. Since the invariant circle is a global attractor, these two solutions coincide up to a time shift: for some $T > 0$ we have

$$\sigma\mathbf{x}(t) = \mathbf{x}(t+T) \quad \text{for all } t;$$

this proves (2.2).

To compete the proof of Theorem 2.2 it remains to show that the equilibria disappear for I exceeding some critical value. The equilibria of the system are obtained by setting the time derivatives to zero in (3.3) and (3.4):

$$\frac{\varepsilon}{N}\sum_{i=1}^{N}\sin(y_i+z+v_{0i})=I \tag{3.24}$$

$$\varepsilon\pi_Y(\sin(\mathbf{y}+z\mathbf{1}+\mathbf{v_0}))-\Delta_Y\mathbf{y}=0. \tag{3.25}$$

Since for sufficiently small ε the function of $\mathbf{y}$ given by

$$F_{z,\varepsilon}(\mathbf{y}):=\varepsilon\pi_Y(\sin(\mathbf{y}+z\mathbf{1}+\mathbf{v_0}))-\Delta_Y\mathbf{y} \tag{3.26}$$

is a small perturbation of the invertible linear map Δ_Y on Y, the implicit function theorem implies that there exists $\varepsilon_0'>0$ such that for any $0<\varepsilon<\varepsilon_0'$ and for any $z\in\mathbb{R}$ there exists a unique $\mathbf{y_0}=\mathbf{y_0}(z,\varepsilon)\in Y$ ($\mathbf{y_0}$ depends smoothly on z and ε) such that $F_{z,\varepsilon}(\mathbf{y_0}(z,\varepsilon))=0$, i.e., $\mathbf{y}=\mathbf{y_0}(z,\varepsilon)$ satisfies Eq. (3.25), or equivalently

$$\varepsilon\sin(\mathbf{y_0}(z,\varepsilon)+z\mathbf{1}+\mathbf{v_0})-\Delta_Y\mathbf{y_0}(z,\varepsilon)=I_N(z,\varepsilon)\mathbf{1}, \tag{3.27}$$

where

$$I_N(z,\varepsilon)=\frac{\varepsilon}{N}\sum_{i=1}^{N}\sin\left(y_{0i}(z,\varepsilon)+z+\frac{2\pi}{N}(i-1)\right). \tag{3.28}$$

We define $\varepsilon_1:=\min\{\varepsilon_0,\varepsilon_0'\}$, and let $\varepsilon\in(0,\varepsilon_1)$. Now for $(z,\mathbf{y})=(z,\mathbf{y_0}(z,\varepsilon))$ to be an equilibrium, we must choose z so that (3.24) holds, or equivalently, in the notation (3.28):

$$I_N(z,\varepsilon)=I.$$

To summarize, the existence of equilibria reduces to a single scalar equation for z. Let now $I_N(\varepsilon):=\sup\{I_N(z,\varepsilon),z\in\mathbb{R}\}$. It can be easily checked that $I_N(\varepsilon)\le\varepsilon$ (computational evidence suggests that, in fact, $I_N(\varepsilon)=C_N\varepsilon^N+O(\varepsilon^{N+1})$ with $C_N>0$). For $I>I_N(\varepsilon)$ there are no equilibria, and thus all solutions approach a traveling wave solution. □

Proof of Theorem 2.3. We must study the equilibria of (2.1), or equivalently, of (3.2) for $N=3$. Since $\sum_{i=1}^{3}y_i=0$ according to (3.1), equilibria of (3.2) satisfy

$$\begin{cases}\varepsilon\sin(y_1+z)=-3y_1+I\\ \varepsilon\sin\left(y_2+z+\dfrac{2\pi}{3}\right)=-3y_2+I\\ \varepsilon\sin\left(-(y_1+y_2)+z+\dfrac{4\pi}{3}\right)=3(y_1+y_2)+I.\end{cases} \tag{3.29}$$

Adding up Eqs. (3.29) gives $I=O(\varepsilon)$, and as a result

$$y_k=O(\varepsilon),\quad k=1,2,3. \tag{3.30}$$

Again adding up Eqs. (3.29) and noting that $\sum_{i=1}^{3}\sin(z+(i-1)\frac{2\pi}{3})=0$, we have:

$$\begin{aligned}I&=\frac{\varepsilon}{3}\sum_{i=1}^{3}\sin\left(y_i+z+(i-1)\frac{2\pi}{3}\right)\\ &=\frac{\varepsilon}{3}\left[\underbrace{\sum_{i=1}^{3}\sin\left(z+(i-1)\frac{2\pi}{3}\right)}_{=0}+O(|y_1|+|y_2|)\right]\\ &=O(\varepsilon^2).\end{aligned}$$

Using this and (3.30) in (3.29) we have:

$$y_1=\frac{-\varepsilon}{3}\sin z+O(\varepsilon^2), \tag{3.31}$$

$$y_2=\frac{-\varepsilon}{3}\sin\left(z+\frac{2\pi}{3}\right)+O(\varepsilon^2). \tag{3.32}$$

Adding up Eqs. (3.29) again, substituting (3.31) and (3.32), and keeping the first two terms of the Taylor expansion, we discover that $O(\varepsilon^2)$ terms cancel:

$$\begin{aligned}I&=\frac{\varepsilon}{3}\left\{\left[\sin z+\cos z\left(\frac{-\varepsilon}{3}\sin z\right)\right]\right.\\ &\quad+\left[\sin\left(z+\frac{2\pi}{3}\right)+\cos\left(z+\frac{2\pi}{3}\right)\left(\frac{-\varepsilon}{3}\sin\left(z+\frac{2\pi}{3}\right)\right)\right]\\ &\quad+\left[\sin\left(z+\frac{4\pi}{3}\right)+\cos\left(z+\frac{4\pi}{3}\right)\right.\\ &\quad\left.\left.\times\left[\frac{\varepsilon}{3}\left(\sin z+\sin\left(z+\frac{2\pi}{3}\right)\right)\right]\right]\right\}+O(\varepsilon^3)\\ &=\underbrace{\frac{\varepsilon}{3}\sum_{i=1}^{3}\sin\left(z+(i-1)\frac{2\pi}{3}\right)}_{=0}-\frac{\varepsilon^2}{9}\sum_{i=1}^{3}\left[\cos\left(z+(i-1)\frac{2\pi}{3}\right)\right.\\ &\quad\left.\times\sin\left(z+(i-1)\frac{2\pi}{3}\right)\right]+O(\varepsilon^3)\\ &=\frac{-\varepsilon^2}{18}\underbrace{\sum_{i=1}^{3}\sin\left(2z+(i-1)\frac{2\pi}{3}\right)}_{=0}+O(\varepsilon^3)\\ &=O(\varepsilon^3).\end{aligned}$$

We thus need to consider $O(\varepsilon^3)$-terms. Adding up Eqs. (3.29) again, and substituting (3.31) and (3.32) gives:

$$I_3(z,\varepsilon)=\frac{-\varepsilon^3}{54}\left(\sum_{i=1}^{3}\sin^3\left(z+(i-1)\frac{2\pi}{3}\right)\right)+O(\varepsilon^4). \tag{3.33}$$

Using the identity

$$\sum_{i=1}^{3}\sin^3\left(z+(i-1)\frac{2\pi}{3}\right)=\frac{-3}{4}\sin 3z, \tag{3.34}$$

proven in the next paragraph, we obtain:

$$I_3(z,\varepsilon)=\frac{\varepsilon^3}{72}\sin 3z+O(\varepsilon^4). \tag{3.35}$$

Thus for $N=3$ the equilibria Eqs. (3.24) and (3.25) reduce to

$$I=I_3(z,\varepsilon)=\frac{\varepsilon^3}{72}\sin 3z+O(\varepsilon^4). \tag{3.36}$$

Let $I_3(\varepsilon):=\sup\{I_3(z,\varepsilon),z\in\mathbb{R}\}$. According to Theorem 2.2 the system undergoes a saddle–node bifurcation at $I=I_3(\varepsilon)$.

Proof of identity (3.34). Note that $\mathrm{Im}(e^{3zi})=\mathrm{Im}(\cos z+i\sin z)^3=3\sin z-4\sin^3 z$. Using this identity by shifting z by multiples of $2\pi/3$ we get

$$\begin{aligned}\sum_{k=0}^{2}\mathrm{Im}\left(e^{3i\left(z+k\frac{2\pi}{3}\right)}\right)&=3\underbrace{\sum_{k=0}^{2}\sin\left(z+k\frac{2\pi}{3}\right)}_{=0}\\ &\quad-4\sum_{k=0}^{2}\sin^3\left(z+k\frac{2\pi}{3}\right)\\ &=-4\sum_{k=0}^{2}\sin^3\left(z+k\frac{2\pi}{3}\right).\end{aligned}$$

On the other hand, $\sum_{k=0}^{2}\mathrm{Im}(e^{3i(z+k\frac{2\pi}{3})})=3\,\mathrm{Im}(e^{3zi})=3\sin 3z$. Comparing the last two results gives (3.34). □

4. Conclusion

In this paper, we proved the existence of traveling wave solutions for the periodic lattice of coupled pendula when the gravity ε is less than a certain constant, or, equivalently, if the coupling is sufficiently large. It should be pointed out that in certain ranges of ε and I the system does exhibit chaotic behavior. Since our proofs are essentially constructive, explicit rigorous bounds guaranteeing a unique globally stable traveling wave can be obtained, in a way similar to how it was done for a simpler case in [27]. Future directions of this work include exploring the estimate on the critical saddle–node torque for the sinusoidal potential for a general N, as well as the more general periodic potentials, and the relationship of this question to other similar phenomena described in [24–26] such as the degree of contact between the boundaries of resonance zones for Mathieu-type equations or between Arnold tongues in circle maps.

Acknowledgment

This work was supported by the NSF grant DMS-0605878.

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