Measuring Curvature with a Bike

MATHEMATICAL CURIOSITIES
By Mark Levi

The following nice fact can be found in [1] (in a slightly different formulation than here): *The center C of curvature of a bike's rear track (see Figure 1) lies at the intersection point of the two axles' extensions, i.e., of the normals to the front and rear tracks.*

I speak here of a mathematician's bike, namely of a fixed length segment *RF* whose front *F* moves along a prescribed path and whose rear *R* has velocity vector constrained to the line *RF*.

[1] posed the problem of finding a *geometrical* proof for this neat fact. I offer such a proof/explanation here, along with a few additional observations.

To see without calculation why this curious fact holds, imagine first locking the steering angle α to a fixed value, as in Figure 2. With the steering locked, the wheels will trace out two concentric circles with the center at the intersection point of the two axles, thus proving/explaining[1] the claim for $\alpha = \text{const.}$

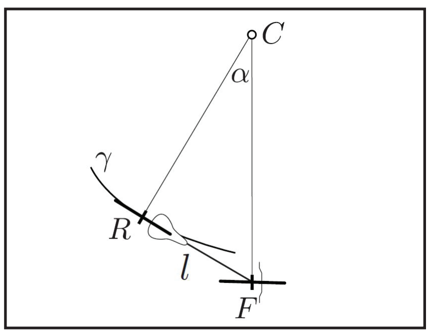

Figure 1.

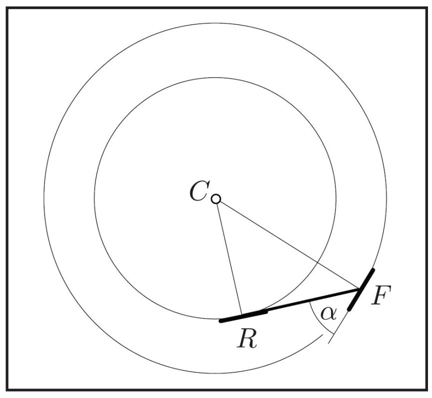

Figure 2.

It remains to remove the constancy assumption, i.e., to explain why curvature κ does not in fact depend on the variation of α but only on α itself (and on the length $l = RF$).

Referring to Figure 3, where *R* moves with speed 1 (treating the arc length *s* as the time), we have

$$\kappa = \frac{d\theta}{ds} = \frac{v \sin\alpha}{l} = \frac{\tan\alpha}{l}, \quad (1)$$

proving the independence of κ on $d\alpha/ds$ and thus justifying the original claim. Actually, the claim also follows directly from (1), which yields $\kappa^{-1} = l\cot\alpha$ and coincides with $RC = l\cot\alpha$ from Figure 1.

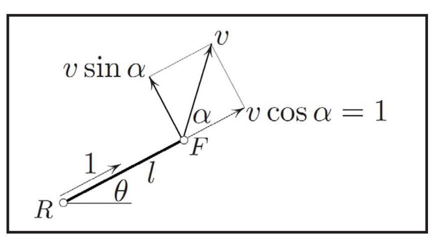

Figure 3. *Proving (1) by applying "$\omega = v/r$" to compute the angular velocity of RF, i.e., the curvature at R.*

I was planning to stop here while writing this note, but then began to wonder if there a way to see the curvature κ itself, rather than the radius of curvature $RC = 1/\kappa$. Figure 4 answers this question:

$$\kappa = KR,$$

assuming $l = 1$ for simplicity. Indeed, the triangles ΔFRC and ΔKRF are similar, so that (taking $l = RF = 1$),

$$\frac{KR}{1} = \frac{1}{RC}, \quad \text{i.e.,}$$

$$KR \cdot RC = 1.$$

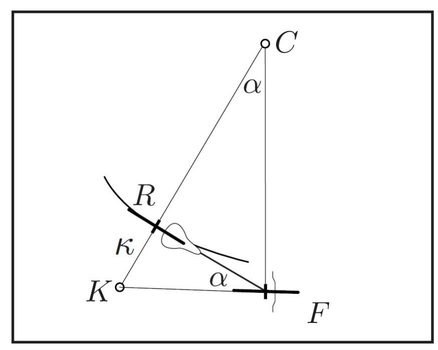

Figure 4. $\kappa = KR$, *where K is the intersection point of the rear axle with the direction line of the front wheel. Here,* $l = RF = 1$.

Thus, $KR = RC^{-1} = (\kappa^{-1})^{-1} = \kappa$, as claimed.

Eyeballing κ while riding the bike would force one to look backwards (and with *K* passing from one side of the line *RF* to the other, doing so becomes particularly difficult and embarrassing). Here is a safer way, which avoids twisting the neck (or breaking it, if v is large). Imagine mounting a light, or better, a laser pointer on the handlebar;

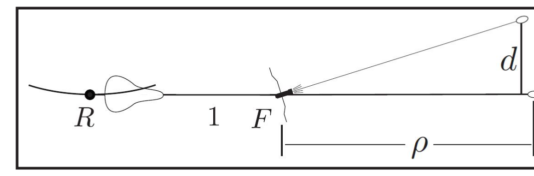

Figure 5. *Measuring the curvature of the rear track with the bike light.*

then the deviation d of the light spot cast on the ground (see Figure 5) determines κ (for $l = 1$) via

$$\kappa = d/\rho + O(d^3).$$

From now on I will probably always think of κ when riding a bike at night.

The figures in this article were provided by the author.

References

[1] Alexander, J.C. (1984). On the Motion of a Trailer-Truck. *SIAM Review, 26*(4), 579-580.

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[1] Proofs which also explain *why* probably deserve a special name, something like "exproof."