Contemporary Mathematics
Volume **698**, 2017
http://dx.doi.org/10.1090/conm/698/14029

Bouncing in gravitational field

Gregory Galperin and Mark Levi

Dedicated to the memory of Kolya Chernov

1. Introduction

In this note we use a variational argument to prove the existence of certain "random" trajectories of a billiard ball bouncing in gravitational field off of a set of obstacles. Henon considered the motions of a ball in gravitational field in the plane, bouncing off of two circles, and studied the set of trajectories encoded by their itineraries. In this note we consider the spatial case: the point mass, subject to gravity acting along the negative z–axis, undergoes repeated collisions with an array of obstacles, as in Figure 1. We will show that under mild assumptions any prescribed itinerary is realizable.

The methods of hyperbolic dynamics have been highly developed, both in the abstract setting and for the more specific application to billiards. However, since we do not insist on the convexity of the reflecting obstacles, hyperbolic methods do not apply; here we use a variational approach instead. Variational methods were used on many occasions, e.g. in [**1**], [**10**], [**6**], [**9**], [**5**], [**3**], [**4**], [**11**].

It should be pointed out that the trajectories we construct are not local action–minimizers (like they are in the Aubry–Mather theory), see Section 5.

2. Statement of the result

We consider a point mass in a gravitational field with constant acceleration g, in the direction of the negative z–axis. When impacting a surface the particle undergoes a perfectly elastic reflection, Figure 1. We will consider a particle bouncing off of infinite family of disjoint surfaces $\Sigma_k \subset \mathbb{R}^3$, $k \in \mathbb{Z}$, such as in Figure 1. We will call these surfaces the *reflectors*. One can think of an array of disjoint spheres, but we consider a much more general infinite array of reflectors satisfying the following three conditions. Loosely speaking, these conditions amount to saying that the reflectors are like islands in a sea which satisfy three conditions: **1.** bounded height and slope; **2.** sufficiently steep beaches, and **3.** none of the islands is too isolated. More precisely, we impose the following conditions.

1. *Bounded height and flatness:* Each reflector Σ_k, $k \in \mathbb{Z}$, is the graph of a smooth function $f_k : D_k \mapsto \mathbb{R}$, where $D_k \in \mathbb{R}^2$ are disjoint closed

The second author gratefully acknowledges the support by the NSF grant DMS-0605878.

topological disks in the plane $z = 0$, with smooth boundaries, satisfying the height condition

$$0 \leq f_k(x, y) \leq 1 \tag{1}$$

and the bounded steepness condition

$$|\nabla f_k| \leq \alpha \tag{2}$$

for some $\alpha > 0$ independent of k and satisfying

$$f_k|_{\partial D_k} = 0 \quad \text{for all} \quad k. \tag{3}$$

2. *Steepness at the boundaries:* There exists $\beta > 0$ such that for all $k \in \mathbb{Z}$ the sine of the angle between the tangent plane to the graph at every boundary point of the reflector and the horizontal exceeds β, or, equivalently, that

$$\mathbf{e}_z \cdot \mathbf{u} \geq \beta \quad \text{on} \quad \partial D_k, \tag{4}$$

where $\mathbf{e}_z$ is the unit vector of the (downward) z–axis and $\mathbf{u}$ is the unit vector tangent to the graph of f_k and normal to the boundary level curve $\{f_k = 0\} = \partial D_k$, Figure 2.[1]

3. *Proximity:* There exists $d > 0$ such that for any reflector Σ there exists another reflector Σ' which is d–close to Σ in the sense that distmax $(\Sigma, \Sigma') < d$.

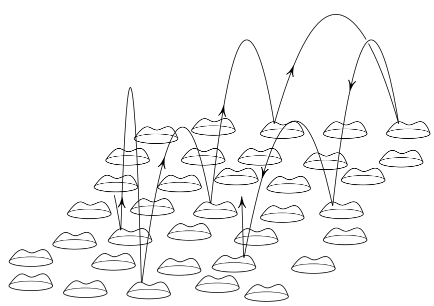

FIGURE 1. A random sequence of collisions

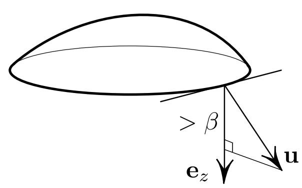

FIGURE 2. Steepness near the boundary

The main result states that any bouncing itinerary is realizable, subject to certain mild assumptions. More precisely, the following theorem holds.

[1]Explicitly, $\mathbf{u}$ is the unit vector parallel to $\langle \nabla f, |\nabla f|^2 \rangle \in \mathbb{R}^3$, i.e. $\mathbf{u} = \frac{1}{|\nabla f|\sqrt{1+|\nabla f|^2}} \langle \nabla f, |\nabla f|^2 \rangle$.

Theorem. *Assume that the reflectors satisfy conditions 1-3 above. Fix any $M > d$, and consider any sequence of reflectors*

$$\cdots, \Sigma_{k_{-1}}, \Sigma_{k_0}, \Sigma_{k_1}, \Sigma_{k_2}, \cdots \tag{5}$$

with $\operatorname{distmax}(\Sigma_{k_i}, \Sigma_{k_{i+1}}) < M$*. Then there exists a gravitational billiard trajectory γ with the itinerary (5) in the sense that the projectile hits the reflectors in the order given by (5). Moreover, such a trajectory exists for any prescribed energy E provided that $E/M > c$, where c is a constant depending only on g, α, β, d.*

Remark 1. The reflectors need not be congruent and need not be convex.

Remark 2. The array of the reflectors need not be periodic.

Remark 3. We prove the existence of an orbit by a variational argument, using a minimization procedure. This may create an impression that the resulting orbits are action–minimizers, i.e. that small perturbations of the trajectory cannot decrease the action. In fact, this is not the case (as explained in Section 5). In fact, the Morse index of the trajectory given by the above theorem is infinite. This is in contrast to other examples where variational method gives local action–minimizers, such as Mather's orbits [**6**], the geodesics in Hedlund's example [**1**, **4**] and the action–minimizing orbits in Arnold diffusion [**2**, **11**].

Some heuristic remarks. Standard tools of hyperbolic dynamics (see, e.g., [**7**]) do not apply without more specific knowledge of the shape the reflectors – a knowledge we don't possess/need. The reason for this failure is the fact that for a non convex reflector such a "camelback" in Figure 1, the "reflection map" (which assigns the incoming tangent vector on one reflector to the incoming tangent vector on the next reflector) can be rather wild. In other words, a certain section map in the phase space does not have good "hyperbolic" properties.

Variational approach, on the other hand, relies only on the rather crude knowledge of the reflector's shape and on the nice dependence of a trajectory connecting two reflectors on the endpoints. This approach deals with the configuration space rather than the phase space.

The phase–space–based proof, had it worked, would go along the following lines. The set of initial data for trajectories leaving a reflector Σ_k is a 4–dimensional cube C_k, represented by a square in Figure 3. One would then consider the *subset* V of initial data from C_k of the trajectories that hit some other reflectors (including C_k itself); the reflected data of the bounced off trajectories form a subset H of initial data in a collection of 4–cubes C_j corresponding to these other reflectors. In other words, we have a mapping from a subset of each C_k into collection of other cubes – a standard setting for a horseshoe map (in $\mathbb{R}^4$). Had the reflectors been convex, one could in fact prove that V and H are unions of "vertical" and "horizontal" strips, and that the map restricted to V satisfies the Conley–Moser cone conditions introduced in [**8**] (for the case of $\mathbb{R}^2$), implying the conjugacy of the "collision map" with a subshift on a space of sequences. This would prove the existence of trajectories for any admissible itinerary, and moreover show their hyperbolicity. Without the convexity assumptions, however, all these statements are unprovable. For instance, the topological conjugacy to the subshift of finite type fails: if one of the reflectors has two sufficiently prominent "humps", then the trajectory can visit the reflector by hitting either of these two humps, thus violating the one-to-one correspondence between the itineraries and the trajectories.

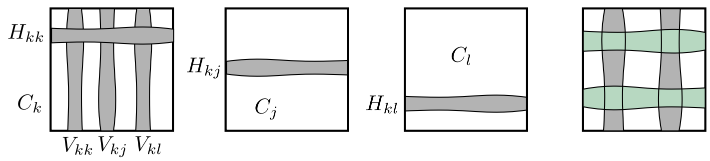

FIGURE 3. A schematic representation of the horseshoe in $\mathbb{R}^4$.

3. Three Lemmas

The proof of the theorem relies on the following three lemmas and is given after the statements and the proofs of these lemmas.

3.1. Lemma 1. The first lemma states that any two reflectors are connected by a "steep" trajectory, provided that the reflectors are not too far apart and that the energy is sufficiently large. More precisely, we have the following.

LEMMA 1. *Fix $M > 0$ and let two reflectors Σ_1, Σ_2 satisfy assumptions 1-3 and* $\operatorname{distmax}(\Sigma_1, \Sigma_2) \leq M$. *Then there exists an energy threshold $E_0 = E_0(M, \alpha)$ such that for any $E > E_0$ and for any pair of points $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ there exists a parabolic trajectory γ with energy E connecting σ_1, σ_2, intersecting Σ_1, Σ_2 transversally and depending continuously on the endpoints σ_1, σ_2. Moreover, the tangent vectors to γ at σ_1, σ_2 can be made arbitrarily close to vertical by choosing sufficiently large energy E: given any $\varepsilon > 0$ there exists $E_1 = E_1(M, \varepsilon)$ such that the downward tangent vectors $\mathbf{T}_i$ (Figure 4) at the endpoints satisfy*

$$\angle(\mathbf{T}_i, \mathbf{e}_z) < \varepsilon, \quad i = 1, 2. \tag{6}$$

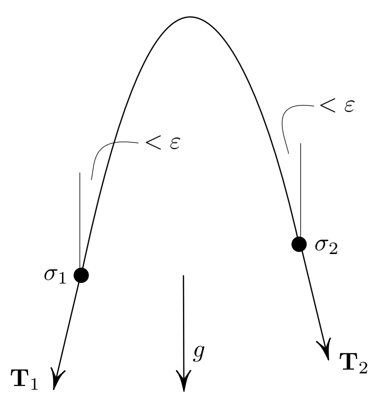

FIGURE 4. Lemma 1.

REMARK 4. *The trajectory given by this lemma is not action–minimizing; rather, its Morse index is* 1, *as explained in* [**4**], *page 178, Figure 5. Briefly, this follows from the fact that there exists a point $\sigma^* \in \gamma$ which is conjugate to σ_1 – namely, σ^* is the point of tangency of γ with the "safety dome" paraboloid (the latter being the envelope of trajectories of the same energy emanating from σ_1). More on this is stated in Section* 5.

Proof of Lemma 1. Rather than using an explicit solution, we choose a more conceptual proof. We start by picking any two points $\sigma_i = (x_i, y_i, z_i)$, $i = 1, 2$, with $|\sigma_2 - \sigma_1| \leq M$, and with $0 \leq z_i \leq 1$. Figure 5 shows the set $W_{v_0}(T)$ consisting of all positions at time T of projectiles launched at time $t = 0$ from point σ_1 with initial speed v_0. The idea of the proof is to show that as the front $W_{v_0}(T)$ expands (as time T grows, with the speed v_0 fixed), it will cross σ_2, provided we choose v_0 sufficiently large. Note that $W_{v_0}(T)$ is a sphere of radius $R = v_0 T$ centered at

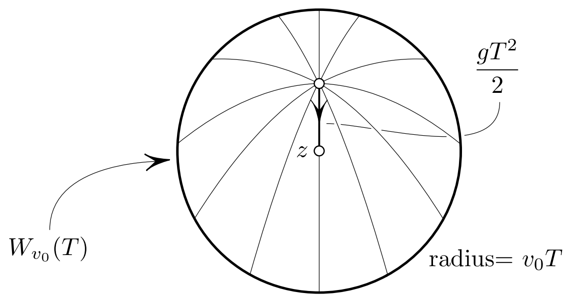

FIGURE 5. The front $W_{v_0}(T)$ is a sphere.

$z = -gT^2/2$, Figure 5. This shows that for

$$T_0 = \frac{2v_0}{g}$$

(the time it takes a projectile launched vertically upwards to return to σ_1), the sphere $W_{v_0}(T_0)$ contains the launch point σ_1, Figure 6. Moreover, if v_0 is large, then the radius $R = v_0 T_0$ is large, and thus the patch $S(T) = W_{v_0}(T) \cap B_{\sigma_1}(M)$ is well approximated by the horizontal plane through σ_1, for all T satisfying $|T - T_0| \leq 1$, Figure 6. Moreover, $S(T)$ has normal speed $O(T)$ (as T changes) for $|T - T_0| \leq 1$; and since σ_2 lies in $B_{\sigma_1}(M)$ with both σ_1, σ_2 lying in the strip $0 \leq z \leq 1$, this implies that for some $T' = T_0 + O(T_0^{-1})$ the surface $W_{v_0}(T')$ contains σ_2, thus proving the existence of the desired trajectory. And the flight from σ_1 to σ_2 is unobstructed by any other reflector. Indeed,

(i) the trajectory is arbitrarily steep in $0 \leq z \leq 1$ if E is sufficiently large;
(ii) reflectors are not too steep by the assumption (2); and
(iii) the disks D_k over which the reflectors lie are disjoint, and the boundaries of the reflectors lie on the same level ($z = 0$) by (3).

Now (i)–(iii) imply that the parabolic flight from σ_1 to σ_2 does not intersect any reflectors during the flight.

To show that this trajectory depends smoothly on the endpoints, we note that it is a geodesic in the Jacobi metric $d\rho = \sqrt{E - mgz}\, ds$ (where ds is the arc length element), and that the point σ_2 is not conjugate to σ_1. The non–conjugacy is seen from the fact that the set of points conjugate to σ_1 is the "safety paraboloid" (written here with σ_1 taken as the origin)

$$z_c = H - \frac{x^2 + y^2}{4H}, \quad H = \frac{gT'^2}{2} \, .$$

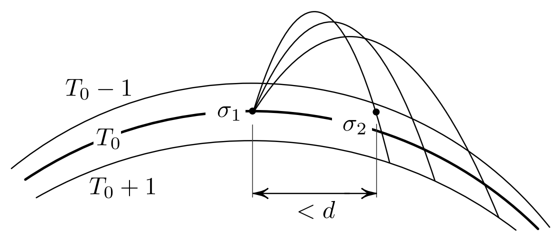

FIGURE 6. "Wave fronts" emanating from σ_1; one of these fronts crosses σ_2.

For large H (caused by large E), conjugate points to σ_1 are high up for $x^2+y^2 < M^2$, and since $z_2 - z_1$ is bounded, σ_2 is not conjugate to σ_1.

It remains to prove the estimate (6). The horizontal velocity of our trajectory is less than the horizontal distance divided by the time, i.e. $v_{\text{horizontal}} < M/T'$, small for T' large (and T *is* large for large E). On the other hand,

$$v_0 = \sqrt{v_{\text{horizontal}}^2 + v_{\text{vertical}}^2}$$

is large, implying that v_{vertical} is large; this explains (6), and completes the proof of Lemma 1. $\diamondsuit$

There is a simple compass and straightedge-type construction of the parabolic trajectory connecting σ_1 and σ_2.

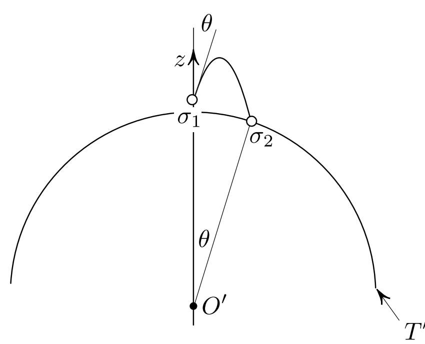

FIGURE 7. Mapping from the velocity sphere to the point on the wave front.

Here is this construction. The magnitude of the velocity v_0 is determined by $mv^2/2 + mgz_1 = E$ (where z_1 is the vertical coordinate of σ_1) and we have (only) the freedom to vary the direction of $\mathbf{v}_0$, hoping to hit σ_2. We thus have a mapping φ which takes a point $\mathbf{v}_0$ on the velocity sphere $|\mathbf{v}_0| = 2(E - mgz_1)$ to the point $\mathbf{x}(T', \mathbf{v}_0)$. Referring to Figure 7, this mapping can be viewed geometrically: the desired image of $\mathbf{v}_0$ is simply the point on the sphere $W_{v_0}(T')$ whose radius is parallel to the vector $\mathbf{v}_0$. This is a consequence of the fact that in the free-falling reference frame, the motion of the particle has constant velocity $\mathbf{v}_0$; the center of the sphere in Figure 7 is simply the origin of the falling frame which starts at $t = 0$, at the moment of launch.

Therefore, in order to hit the point σ_2 from point σ_1, we simply choose the vector $\mathbf{v}_0$ parallel to the radius $O'\sigma_2$ of the sphere $W_{v_0}(T')$ that connects its center O' with the point σ_2 belonging to this sphere. The projectile emanated from point σ_1 with the initial velocity $\mathbf{v}_0$ will describe the sought parabolic trajectory γ that intersects transversally (according to Lemma 1) both the surfaces Σ_1 and Σ_2 at the points σ_1 and σ_2, respectively. This geometric construction shows, once again, that the trajectory γ depends smoothly on its endpoints σ_1 and σ_2.

3.2. Lemma 2. Before stating the second lemma, we fix $M > d$, α and β as in (2) and (4), and make an additional assumption that $\sin\alpha > \beta$. We also fix $E > E_0$, where E_0 is the energy threshold given by Lemma 1. For two points $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ we define the action $A(\sigma_1, \sigma_2)$ as the length of the trajectory $\gamma(\sigma_1, \sigma_2)$ in the Jacobi metric:

$$A(\sigma_1, \sigma_2) = \int_{\gamma(\sigma_1,\sigma_2)} v \, d\ell, \;\; v = \sqrt{2(E - mgz)}, \tag{7}$$

where $d\ell$ is an element of Euclidean length; for E sufficiently large there are two trajectories connecting σ_1, σ_2; we choose the one with the higher maximum.[2] The positive sign in (7) is due to the fact that z–axis points downwards.

LEMMA 2. *Fix $M > 0$, and let Σ_-, Σ, and Σ_+ be three reflectors satisfying (2) and (4) such that* dist $(\Sigma_-, \Sigma) \leq M$, dist $(\Sigma, \Sigma_+) \leq M$. *Consider the action*

$$L(\sigma) = A(\sigma_-, \sigma) + A(\sigma, \sigma_+), \tag{8}$$

where we treat $\sigma_\pm \in \Sigma_\pm$ as fixed, and $\sigma \in \Sigma$ as variable. Then, in the assumptions of Lemma 1 and the preceding paragraph, L attains a minimum at $\sigma = \sigma_{min}$ in the interior of Σ; moreover, every minimizer σ_{min} is bounded away from the boundary $\partial\Sigma$ by a constant $\delta > 0$ which depends only on M, α, β and E but not on σ_-, σ_+.[3]

Proof of Lemma 2. As a continuous function on the compact set Σ, the action L achieves a minimum on Σ; to show that the minimum is interior, we will show that the outward directional derivative on $\partial\Sigma$ is positive. We need an expression for the directional derivative of L. First, let $\mathbf{u}$ be a unit tangent vector to Σ at any point σ (not necessarily on the boundary yet). We note that the directional derivative of the action is given by

$$D_{\mathbf{u}} A(\sigma, \sigma_+) = v \mathbf{T}_+ \cdot \mathbf{u}, \tag{9}$$

where $v = \sqrt{2(E - mgz)}$ is the speed of the particle at σ, and where $\mathbf{T}_+$ is the unit tangent vector at σ to the trajectory $\gamma(\sigma, \sigma_+)$, as shown in Figure 8 (by our convention the tangents to $\gamma(\sigma, \sigma_+)$ at the ends of the trajectory are "outward", i.e. they point away from the other end). Differentiating the sum of actions (8) along $\mathbf{u}$ we obtain

$$D_{\mathbf{u}} L(\sigma) \overset{(9)}{=} v \, (\mathbf{T}_- + \mathbf{T}_+) \cdot \mathbf{u}. \tag{10}$$

Now, according to Lemma 1, there exists E_0 such that for all $E \geq E_0$ we have

$$\angle(\mathbf{T}_\pm, \mathbf{e}) \leq \beta, \quad \text{where} \quad \mathbf{e} = -\mathbf{e}_z,$$

[2] In fact, this trajectory is not a minimizer of the action, but rather a saddle; its Morse index is 1, see [**4**].

[3] However, δ may depend on the choice of Σ_-, Σ, Σ_+.

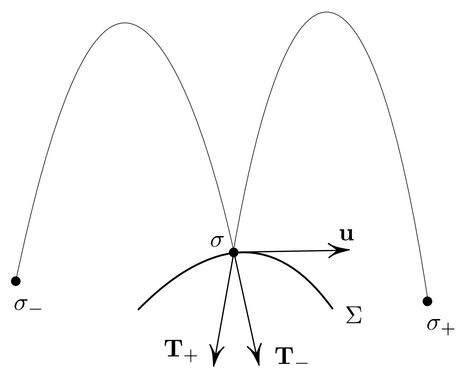

FIGURE 8. Proof of Lemma 2.

and thus

$$\mathbf{T}_{\pm} = \mathbf{e} + \widehat{\boldsymbol{\beta}}, \tag{11}$$

where $\widehat{\boldsymbol{\beta}}$ denotes a vector whose absolute value is less than β. Assume finally that $\sigma \in \partial\Sigma$, so that (4) holds. Substituting (11) into (10), we obtain

$$D_{\mathbf{u}}L(\sigma) = v(2\mathbf{e} + 2\widehat{\boldsymbol{\beta}}) \cdot \mathbf{u} \overset{(4)}{\geq} v(2\beta - 2\widehat{\beta}) > 0.$$

We showed that the minimum of L is not on $\partial\Sigma$, i.e. it is achieved in the interior of Σ. To see that the minimum is *uniformly bounded* away from $\partial\Sigma$, we simply have to repeat the proof with β replaced by $\beta/2$ in (11). This completes the proof of Lemma 2. ♢

3.3. Lemma 3.

LEMMA 3. *If $\sigma_{min} \in \Sigma$ is an interior minimum of L, then $\gamma(\sigma_-, \sigma) \cup \gamma(\sigma, \sigma_+)$ is a true billiard trajectory from σ_- to σ_+ via σ.*

Proof of Lemma 3. According to (10) we have

$$D_{\mathbf{u}}L(\sigma)\big|_{\sigma=\sigma_{min}} = v(\mathbf{T}_- + \mathbf{T}_+) \cdot \mathbf{u} = 0.$$

for any $\mathbf{u} \in T_\sigma\Sigma$. But this is precisely the statement of the collision law. This completes the proof of Lemma 3. ♢

4. Proof of the Theorem

<u>Step 1</u> *(the finite part).* Consider any sequence of reflectors subject to the assumptions of the Theorem:

$$\ldots,\ \Sigma_{-1},\ \Sigma_0,\ \Sigma_1,\ \ldots \tag{12}$$

We first show that for any finite segment

$$\Sigma_{-n},\ \ldots,\ \Sigma_0,\ \ldots,\ \Sigma_n$$

there exists a billiard trajectory which collides with the reflectors in the order listed. Fixing the first and the last points $\sigma_{\pm n} \in \Sigma_{\pm n}$, we consider the function of the intermediate points:

$$L(\sigma_{-n+1}, \ldots, \sigma_{n-1}) = \sum_{k=-n}^{n-1} A(\sigma_k, \sigma_{k+1}) \tag{13}$$

defined on the direct product $\Sigma_{-n+1} \times \ldots \times \Sigma_{n-1}$. This function attains its minimum (which exists by compactness) in the interior of its domain, as follows from Lemma 2. By the same Lemma, the points σ_k of this minimizer are bounded away from the boundary of each Σ_k by some $\delta_k > 0$. By Lemma 3, this minimizer – the union of $2n+1$ parabolas – is a true billiard trajectory undergoing $2n-1$ reflections.

Step 2 *(constructing an infinite sequence).* We will construct the infinite trajectory by a diagonal process. We begin by fixing an infinite pseudo–orbit $\{\sigma_k\}$ with $\sigma_k \in \Sigma_k$. With this orbit fixed, we now vary it: for an integer n, let us now make σ_k with $|k| \leq n$ variable. According to the preceding step, there exists an interior minimum of the action (13) with the finite number of variables; let us denote this minimizer by

$$\mu_n = (\ldots,\ \sigma_{-n},\ \underline{\sigma^{\mathbf{n}}_{\mathbf{-n+1}},\ \sigma^{\mathbf{n}}_{\mathbf{-n+2}},\ \ldots,\ \sigma^{\mathbf{n}}_{\mathbf{0}},\ \ldots,\ \sigma^{\mathbf{n}}_{\mathbf{n-2}},\ \sigma^{\mathbf{n}}_{\mathbf{n-1}}},\ \sigma_n,\ \ldots); \tag{14}$$

we thus minimized only over the entries with $|k| < n$, fixing the others. To reiterate, we constructed a sequence of pseudo–orbits (corresponding to $n = 1, 2, \ldots$)

$$\mu_1 = (\ldots,\ \sigma_{-3},\ \sigma_{-2},\ \sigma_{-1},\ \underline{\sigma^{\mathbf{1}}_{\mathbf{0}}},\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \ldots) \tag{15}$$

$$\mu_2 = (\ldots,\ \sigma_{-3},\ \sigma_{-2},\ \underline{\sigma^{\mathbf{2}}_{\mathbf{-1}},\ \sigma^{\mathbf{2}}_{\mathbf{0}},\ \sigma^{\mathbf{2}}_{\mathbf{1}}},\ \sigma_2,\ \sigma_3,\ \ldots) \tag{16}$$

$$\mu_3 = (\ldots,\ \sigma_{-3},\ \underline{\sigma^{\mathbf{3}}_{\mathbf{-2}},\ \sigma^{\mathbf{3}}_{\mathbf{-1}}, \sigma^{\mathbf{3}}_{\mathbf{0}},\ \sigma^{\mathbf{3}}_{\mathbf{1}},\ \sigma^{\mathbf{3}}_{\mathbf{2}}},\ \sigma_3,\ \ldots) \tag{17}$$

with longer and longer "true cores". We would have liked to show that the sequence

$$\mu_1, \mu_2, \ldots, \mu_n, \ldots \tag{18}$$

of these pseudo–orbits converges coordinate-wise to the true billiard trajectory. This is not generally the case, but we will extract a subsequence that does converge to the true infinite billiard trajectory.

Let us extract a subsequence of (18) for which the $k = 0$ coordinate converges; denote this subsequence by

$$\mu_{11}, \mu_{12}, \mu_{13}, \ldots\ .$$

From this subsequence, in turn, we extract a subsequence for which the coordinates with $|k| < 2$ converge, denoting it by

$$\mu_{21}, \mu_{22}, \mu_{23}, \ldots\ .$$

Proceeding in this way we obtain an infinite table

$$\begin{matrix} \mu_{11}, & \mu_{12}, & \mu_{13}, & \ldots \\ \mu_{21}, & \mu_{22}, & \mu_{23}, & \ldots \\ \mu_{31}, & \mu_{32}, & \mu_{33}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \tag{19}$$

where each row is a subsequence of the preceding one, and where in the kth row each coordinate with $|n| < k$ converges.

For the *diagonal sequence* μ_{nn} every single coordinate converges.

Denote the limiting sequence by

$$\mu_\infty = (\ldots, \sigma_{-1}^\infty, \sigma_0^\infty, \sigma_1^\infty, \ldots)\ .$$

It remains to show that this sequence corresponds to a *billiard trajectory*.

Indeed, in our construction the function

$$L_k^n(\sigma) = A(\sigma_{k-1}^n, \sigma) + A(\sigma, \sigma_{k+1}^n) \qquad (k < n)$$

converges as $n \to \infty$ to the function

$$L_k^\infty(\sigma) = A(\sigma_{k-1}^\infty, \sigma) + A(\sigma, \sigma_{k+1}^\infty).$$

But $L_k^n(\sigma)$ has a minimum at $\sigma = \sigma_k^n$; and since the derivatives of L_k^n are bounded (according to Lemma 2), we conclude that $\sigma_k^\infty = \lim_{n\to\infty} \sigma_k^n$ is a critical point of L_k^∞. This shows that μ_∞ gives a true billiard trajectory and completes the proof of Theorem 1. $\diamondsuit$

5. Local versus global minimizers

We constructed a trajectory with a prescribed itinerary by minimizing the discretized action (7). This may create an impression that the trajectory is a local minimizer, i.e. that its Maupertuis length is no greater than that of any of its small perturbations. This, however, is not the case, as we now show. The key is that the trajectory given by Lemma 1 – the steep parabola connecting two given points σ_1 and σ_2 – is not the minimizer of the Maupertuis length, but rather a saddle point with Morse index 1, as explained in the remark after Lemma 1. In fact, one can show that the Morse index of every trajectory given by our theorem is infinite. This follows from the fact that there are infinitely many intervals between collisions, and that one can make action–*decreasing* perturbations on each such interval *independently* of one another.

References

[1] Gustav A. Hedlund, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*, Ann. of Math. (2) **33** (1932), no. 4, 719–739, DOI 10.2307/1968215. MR1503086

[2] Vadim Kaloshin and Mark Levi, *Geometry of Arnold diffusion*, SIAM Rev. **50** (2008), no. 4, 702–720, DOI 10.1137/070703235. MR2460801

[3] V. V. Kozlov, *Calculus of variations in the large and classical mechanics* (Russian), Uspekhi Mat. Nauk **40** (1985), no. 2(242), 33–60, 237. MR786086

[4] Mark Levi, *Shadowing property of geodesics in Hedlund's metric*, Ergodic Theory Dynam. Systems **17** (1997), no. 1, 187–203, DOI 10.1017/S0143385797060999. MR1440775

[5] John N. Mather, *Arnold diffusion by variational methods*, Essays in mathematics and its applications, Springer, Heidelberg, 2012, pp. 271–285, DOI 10.1007/978-3-642-28821-0_11. MR2975591

[6] John N. Mather and Giovanni Forni, *Action minimizing orbits in Hamiltonian systems*, Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991), Lecture Notes in Math., vol. 1589, Springer, Berlin, 1994, pp. 92–186, DOI 10.1007/BFb0074076. MR1323222

[7] Jürgen K. Moser, *Lectures on Hamiltonian systems*, Mem. Amer. Math. Soc. No. 81, Amer. Math. Soc., Providence, R.I., 1968. MR0230498

[8] Jürgen Moser, *Stable and random motions in dynamical systems*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1973. With special emphasis on celestial mechanics; Hermann Weyl Lectures, the Institute for Advanced Study, Princeton, N. J; Annals of Mathematics Studies, No. 77. MR0442980

[9] P.Y. LeDaeron S. Aubry, *The Discrete Frenkel-Kontorova Model and Its Extensions*, Physica 8D (1983), 381–422.

[10] H. Seifert, *Periodische Bewegungen mechanischer Systeme* (German), Math. Z. **51** (1948), 197–216, DOI 10.1007/BF01291002. MR0025693

[11] Vadim Kaloshin, Mark Levi, and Maria Saprykina, *Arnol'd diffusion in a pendulum lattice*, Comm. Pure Appl. Math. **67** (2014), no. 5, 748–775, DOI 10.1002/cpa.21509. MR3179666

Department of Mathematics and Computer Science, Eastern Illinois University, Illinois 61920

E-mail address: ggalperin@eiu.edu

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

E-mail address: levi@math.psu.edu