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# Composition of rotations and parallel transport

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**Abstract.** This note provides the details and proofs of the results announced by Levi 1993 *Fields Insitute Communications* vol 1 pp 133–8. The main result of this note is a geometrical representation of the reconstruction problem for SO(3) in terms of parallel transport. It is, of course, well known that the solution of a linear equation $\dot{x} = \Omega(t)x$ in $\mathbf{R}^n$ cannot in general be expressed by

$$x(t) = e^{\int_0^t \Omega(\tau)\,d\tau} x(0) \qquad (false)$$

because the coefficient matrices $\Omega(t_1)$ and $\Omega(t_2)$ may fail to commute for $t_1 \neq t_2$. Nevertheless, when $n = 3$ and when $\Omega(t)$ is skew-symmetric, i.e. when it lies in the Lie algebra of the group of rigid rotations in $\mathbf{R}^3$, the above false formula is almost correct, as we will show here. The main result of this note is a geometrical expression for the matrix solution $X(t)$ of matrix equations on $TSO(3)$ of the form

$$\dot{X} = \Omega(t)X, \quad \Omega^T = -\Omega \qquad (*)$$

where $X$, $\Omega$ are $3 \times 3$ matrices with real coefficients. The argument relies on a theorem of Poinsot together with some observations on geodesic curvatures of moving curves.

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## 1. Composition of rotations, reconstruction problem and equation (*)

*An isomorphism* $\hat{}$ : $\mathbf{R}^3 \to \mathbf{so}(3)$. To a given $\omega \in \mathbf{R}^3$ we associate the skew-symmetric matrix $\hat{\omega} \in so(3)$ via

$$\hat{\omega}v = \omega \times v \quad \text{for all} \quad v \in \mathbf{R}^3.$$

This is the standard identification of the Lie algebra $so(3)$ with $\mathbf{R}^3$. Physically this is the identification of the infinitesimal generator of the rigid rotation with the angular velocity vector. Explicitly,

$$\hat{\omega} \equiv \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \text{where} \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad (1)$$

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Let us denote the inverse of the isomorphism $\hat{}: \mathbf{R}^3 \to \mathbf{so(3)}$ by $\check{}: so(3) \to \mathbf{R}^3$, so that in the above notation $\check{\Omega} = \omega$.

*The composition of a continuous family of rotations* reduces to equation (*) as follows. Let $X(t) \in SO(3)$ describe the orientation of body relative to an inertial frame, let $\omega_B$ be the body angular velocity expressed in the body frame†, and let $\omega = \omega_S$ be the angular velocity in the space (inertial) frame. Then the velocity of a point $a$ of a body in the space frame is given by $(\dot{Xa}) = \dot{X}a = (\dot{X}X^{-1})(Xa) \equiv \Omega_S(Xa) = \omega_S \times (Xa)$, and the definition of the space angular velocity $\dot{X}X^{-1} = \hat{\omega}_S$, or

$$\dot{X} = \hat{\omega}_S X$$

is equivalent to equation (1). Further details on these definitions can be found in most mechanics books, e.g. [A].

*The reconstruction problem* in the rigid body motion deals with recovering the orientation of a rigid body in *space* given the time–evolution of its angular velocity in the *body* frame ([M]). The reconstruction problem is equivalent to equation (*): indeed, the definition of the body angular velocity $X^{-1}\dot{X} = \hat{\omega}_B$ is equivalent to

$$\dot{Y} = -\hat{\omega}_B Y, \qquad Y = X^T.$$

## 2. Results

**Definition**. Let $\mathrm{Rot}_v(\theta)$ denote the rotation around the direction of the unit vector $v$ through the angle $\theta$:

$$\mathrm{Rot}_v(\theta) = \exp(\theta\hat{v}),$$

where $\hat{}$ is the isomorhism defined in the introduction.

**Theorem 1.** *For any matrix equation on $SO(3)$ of the form $\dot{X} = \Omega(t)X$ with $\Omega^T = -\Omega$ and $X(0) = I$ there exist two associated curves $S$ and $B_0$ on the unit sphere (figure 1) defined in section 2.2 below, both parametrized by $t$ and tangent at $t = 0$, such that the solution matrix is given by*

$$X(t) = P_S(t)\mathrm{Rot}_v(\alpha)P_{B_0}^{-1}(t), \tag{2}$$

*where $\alpha = \int_0^t |\omega(\xi)|\,\mathrm{d}\xi$ is the cumulative angle, $v = \frac{\omega(0)}{|\omega(0)|}$ with $\omega = \check{\Omega}$ and where the orthogonal matrices $P_S$ and $P_{B_0}$ are the operators of parallel transport along the curves $S$ and $B_0$, all defined in the next paragraph and in section 2.2.*

† This should not be confused with the velocity of the body relative to its own frame, which is, of course, zero.

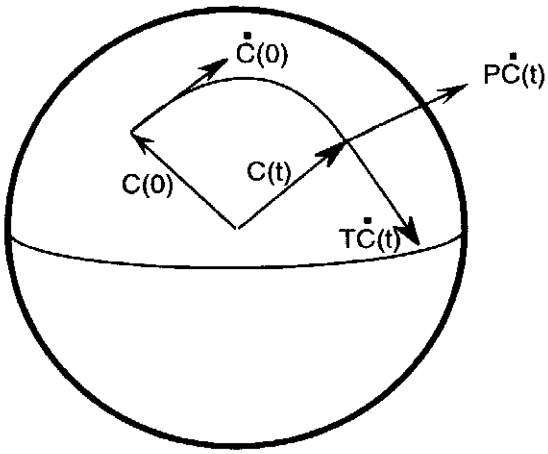


**Figure 1.** The definition of tangent transport.

## 2.1. *Definition of the transport operators*

$P_C(t) \in SO(3)$ of parallel transport from $C(0)$ to $C(t)$ is defined uniquely by the condition that $P_S(t)$ takes tangent vectors to $S^2$ at $C(0)$ into their parallel translates along $C$ at $C(t)$. For future use we also define the tangent transport operator $T_C(t) \in SO(3)$ by the conditions

$$T_C(t)C(0) = C(t), \qquad T_C(t)\frac{\dot{C}(0)}{|\dot{C}(0)|} = \frac{\dot{C}(t)}{|\dot{C}(t)|}.$$

**Remark 1.** Theorem 1 can be thought of as an expression for the product of non-commuting orthogonal matrices (we drop the subscript: $\Omega_S \equiv \Omega$):

$$X(t) = \lim_{N\to\infty} e^{\Omega_N} \cdot e^{\Omega_{N-1}} \cdot \ldots \cdot e^{\Omega_1} = P_S e^{\lim_{N\to\infty}(|\Omega_N|+\ldots+|\Omega_1|)\frac{\Omega_1}{|\Omega_1|}} P_{B_0}^{-1},$$

where $\Omega_n \equiv \Omega_{n,N} = \Omega\big(\frac{(n-1)t}{N}\big)\frac{t}{N}$ and $|\Omega|$ denotes $\frac{1}{\sqrt{2}}$ euclidean norm of $\Omega$. The exponential term in (2) is the rotation around the fixed axis $\omega(0)$ by the amount $\int_0^t |\omega|\,d\xi$, disregarding the reorientation of the axis $\omega(t)$ with time. The factors $P_S$ and $P_{B_0}^{-1}$ account precisely for this reorientation.

For future reference we note the relationship between $P_C(t)$ and $T_C(t)$: let $k_C(t)$ be the geodesic curvature of the curve $C(t)$ on the unit sphere; then the total geodesic curvature

$$K_C(t) = \int_0^t k_C(\xi)|\dot{C}(\xi)|\,d\xi,$$

and

$$T_C(t) = P_C(t)\exp(K_C(t)\hat{C}(0)). \tag{3}$$

To prove equation (3) we note that $T_C(t)$ and $P_C(t)$ both take $C(0)$ into $C(t)$ and so are related by rotation about $\hat{C}(0)$: $T_C(t) = P_C(t)\exp(\theta(t)\hat{C}(0))$. The rotation angle $\theta$ is equal to the angle between the tangent to $C(t)$ and the parallel transport of $\dot{C}(0)$ to the same point. The geodesic curvature at $C(t)$ is $K_C(t) = \mathrm{d}\theta/\mathrm{d}t$ which completes the proof of equation (3).

### *2.2. The curves $S$ and $B_0$*

We will denote by $S(t)$ the path on the unit sphere traced by the line of the (space) angular velocity vector:

$$S(t) = \frac{\omega}{|\omega|}, \qquad \omega = \check{\Omega}.$$

The 'body curve' $B_0$ is, by the definition, traced out by the normalized vectors of the body angular velocity:

$$B_0(t) = X^{-1}(t)S(t),$$

figure 2. We assume that $X(0) = I$, so that $S(0) = B_0(0)$.

*Poinsot's theorem*, after a slight reformulation, [W], states that the curve $B_t$ parametrized by $\tau$:

$$B_t(\tau) = X(t)B_0(\tau)$$

rolls, as $t$ changes, without sliding along the curve $S$, both curves lying on the unit sphere, figure 2.

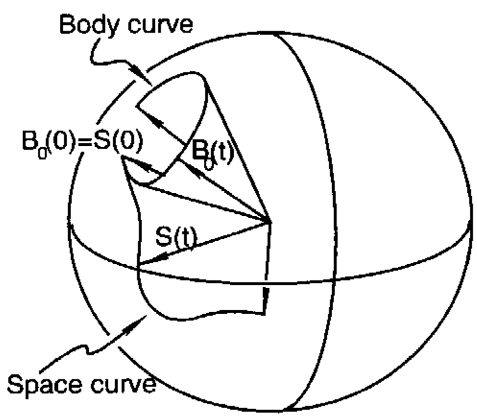


**Figure 2.** Space curve $S$ and the body curve $B_0$.

'Rolls without sliding' means that (1) the two curves are tangent at all times and that (2) the point of the moving curve which is in contact with the stationary curve at a given moment has zero velocity at that moment. More precisely, interpreting $t$ as the time and $\tau$ as the parameter along both curves, the statements (1) and (2) amount to

$$\text{tangency}: \quad \frac{\partial}{\partial \tau} S(\tau) = \frac{\partial}{\partial \tau} B_t(\tau)|_{\tau=t} \tag{4}$$

$$\text{no sliding}: \quad \frac{\partial}{\partial t} B_t(\tau)|_{t=\tau} = 0. \tag{5}$$

For completeness, we give a short proof of Poinsot's theorem in section 3.1.

A natural question arises: what is the relationship between $S(t)$ and $B_0(t)$? The answer is given by the following observation which, besides its intrinsic interest, allows us to compute the body curve $B_0$ without the full knowledge of the matrix $X(t)$ despite the fact that $X(t)$ enters the definition of $B_0$.

**Theorem 2.** *The geodesic curvatures $k_B(\tau)$ and $k_S(\tau)$ of $B_0$ and $S$ are related via*

$$k_B = k_S - \frac{|\omega|}{|\dot{S}|}, \qquad \omega = \check{\Omega}. \tag{6}$$

If we choose $t = s$ to be the arclength, i.e. if $|\dot{S}| = 1$, then we get simply

$$k_B(s) = k_S(s) - |\omega|.$$

The proof is given in section 3.2. This theorem allows us to compute the geodesic curvature of $B_0$ *explicitly*. Finding $B_0$ reduces therefore to the determination of a curve on the sphere from the prescribed geodesic curvature.

**Theorem 3.** *The fundamental solution matrix $X(t)$ of equation (1) factorizes as the product of two tangent transports, figure 2:*

$$X(t) = T_S(t) T_{B_0}^{-1}(t). \tag{7}$$

The results of this paper are related to the earlier theorems of Diliberto [DI], see also the more recent exposition and correction by Chicone [C]. Diliberto dealt with a different question of finding a closed form expression for the Poincaré map of a closed orbit for a planar vector field. The answer he gave was in terms of the curvature of the orbit as well as some quantities computed from the linearized equation, in the same spirit as we have done here for the flow on $SO(3)$. The common thread unifying the two problems is the role of parallel transport and the associated holonomy terms in the final expressions. In our case these terms are given by the matrices $P_S$ and $P_B$ in the main theorem below. This is closely related to Berry's phase. Further examples, discussion and references on the latter can be found in [MMR], [S], [B], [BH], [SW], [M], [L].

## 3. Proofs

We start with the proof of Poinsot's theorem, then we prove the curvature theorem 2 and finally the main theorem 1 (the rotation formula) together with theorem 3.

*3.1. Proof of Poinsot's theorem*

We have to check the tangency and the non-sliding conditions (4) and (5). Differentiating the right–hand side in (4), we obtain

$$\frac{\partial}{\partial\tau}B_t(\tau)_{|\tau=t} = X(t)\left[-X^{-1}(\Omega X)X^{-1}\right]_{\tau=t}S(t) + X(t)X^{-1}(t)\dot{S}(t) = \dot{S}(t),$$

as claimed; here we used the fact that $\Omega(\tau)$ is the instantaneous axis of rotation at $t = \tau$: $\Omega S = \Omega\frac{\check{\Omega}}{|\check{\Omega}|} = \check{\Omega} \times \check{\Omega}/|\check{\Omega}| = 0$.
Proof of (5): $\frac{\partial}{\partial t}B_t(\tau)_{|\tau=t} = \dot{X}(t)X^{-1}(t)S(t) = \Omega S = 0$. □

*3.2. Proof of theorem 2 (the curvature formula (6))*

We use the formula for the geodesic curvature of a curve $r(t)$ on the unit sphere:

$$k = \frac{1}{|\dot{r}|^2}r \times \dot{r} \cdot \left(\frac{\dot{r}}{|\dot{r}|}\right)^{\cdot}.$$

To compute the geodesic curvature of $B_0$ we substitute $B_0(\tau) = X^{-1}(\tau)S(\tau)$ into the formula above, and recalling that $\dot{X} = \Omega X$, $\Omega v = \check{\Omega} \times v$ and $\Omega(t)S(t) \equiv 0$, we obtain after some simplificaton:

$$k_B = \frac{1}{|\dot{S}|^2}S \times \dot{S} \cdot \left(\frac{\dot{S}}{|\dot{S}|}\right)^{\cdot} - \frac{1}{|\dot{S}|^3}(S \times \dot{S}) \cdot (\Omega\dot{S}) = k_S - \frac{|\omega|}{|\dot{S}|}. \tag{8}$$

We used the orthogonality of $X$ and the facts that $\Omega\dot{S} = \check{\Omega} \times \dot{S}$ and $S = \frac{\check{\Omega}}{|\check{\Omega}|}$. This proves theorem 2. □

*3.3. Proof of theorem 3 (equation (7))*

It suffices to show that

$$T_S(t)T_{B_0}^{-1}(t)v = X(t)v \tag{9}$$

for two noncollinear choices of $v \in \mathbf{R}^3$. Let us first take $v_1 = B_0(t)$ as one such choice; we get $T_S(t)T_{B_0}^{-1}(t)B_0(t) \stackrel{A}{=} T_S(t)B_0(0) \stackrel{B}{=} T_S(t)S(0) \stackrel{C}{=} S(t) \stackrel{D}{=} X(t)B_0(t)$; $A$ and $C$ hold by the definition of the tangent transport, while $B$ and $D$ hold by the definition of $B_0(t) = X^{-1}(t)S(t)$. The proof of (10) for $v = \dot{B}_0(t)$, namely, $T_S(t)T_{B_0}^{-1}(t)\dot{B}_0(t) = X(t)\dot{B}_0(t)$, is the same. □

*3.4. Proof of the main theorem*

The decomposition (3) gives us

$$T_S(t) = P_S(t)\mathrm{e}^{K_S(t)\hat{S}(0)}, \qquad T_{B_0}(t) = P_{B_0}(t)\mathrm{e}^{K_{B_0}(t)\hat{S}(0)}$$

and the factorization formula (7) gives

$$X(t) = P_S(t)\mathrm{e}^{(K_S(t)-K_{B_0}(t))\hat{S}(0)}P_{B_0}^{-1}(t); \tag{10}$$

by the curvature theorem (6) we get

$$K_S(t) - K_{B_0}(t) = \int_0^t (k_S(\xi) - k_{B_0}(\xi))|\dot{S}(\xi)|\,\mathrm{d}\xi = \int_0^t |\omega(\xi)|\,\mathrm{d}\xi.$$

Substituting this into (10) and using $\hat{S}(0) = \frac{\omega(0)}{|\omega(0)|}$, $\omega = \check{\Omega}$ we obtain the rotation formula (2). □

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