Symmetry and Polar Decomposition by Mechanics

MATHEMATICAL CURIOSITIES

By Mark Levi

There are several mechanical or geometrical interpretations of the symmetry of a matrix; I would like to describe one that recently occurred to me. It is likely that others have thought of it before, though I did not do a literature search to confirm.

Let us interpret the square $n\times n$ matrix A as a frame of its column vectors $\boldsymbol{a}_k\in\mathbb{R}^n$, thought of as n rigid rods that are welded together and pivot on the origin O. As illustrated in Figure 1, let us connect the tip of the kth column/rod $\boldsymbol{a}_k$ to the tip of the coordinate unit vector $\boldsymbol{e}_k$ by a Hookean spring, i.e., the spring whose tension is directly proportional to its length. All springs have the same Hooke's constant.

Claim 1: *An $n\times n$ $(n\geq 2)$ matrix is symmetric if and only if the aforementioned mechanical system is in equilibrium.*

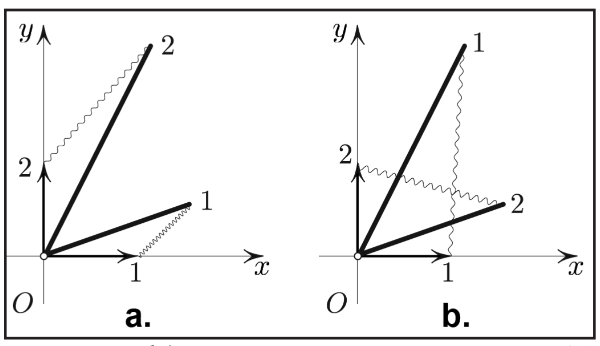

Figure 1. *The kth column vector $\boldsymbol{a}_k$ is connected to the kth coordinate unit vector $\boldsymbol{e}_k$.* **2a.** $\det A>0$. **2b.** $\det A<0$.

Indeed, in an equilibrium state the torque around O in any ij-coordinate plane vanishes; this amounts to $a_{ij}-a_{ji}=0$.

Claim 2: *A matrix is positive-definite if and only if the frame is right-handed and in a stable equilibrium.*

The proof of this claim in dimension 2 is almost purely geometrical. Figure 2 illustrates stable equilibria with the positively oriented frames. No combination of quadrants other than those in Figure 2 can occur.

It is clear that both eigenvalues are positive in either case. Indeed, with QN denoting the Nth quadrant, we have (see Figure 3)

$$A(Q1)\subset Q1 \quad\text{and}\quad A(Q2)\supset Q2$$

for matrix A in Figure 2a. We use the fact that the matrix maps the basis $\boldsymbol{e}_1,\boldsymbol{e}_2$ to the frame $\boldsymbol{a}_1,\boldsymbol{a}_2$. According to a fixed point theorem, $Q1$ and $Q2$ both contain eigenvectors with positive eigenvalues. We treat the matrix in Figure 2b similarly, with the same conclusion of two positive eigendirections. By contrast, Figure 1b depicts a negatively oriented frame, and we have $A(Q2)\supset Q4=-Q2$ for the corresponding matrix A, implying the existence of a negative eigenvalue.

Connection to the Toeplitz Norm

Potential energy of the system in Figure 1 is a mechanical interpretation of the Toeplitz norm $\|A-I\|^2$, up to a constant factor depending on Hooke's constant. We recall that the Toeplitz norm of a square matrix X is defined as the root of the sum of its elements' squares: $\|X\|^2=\mathrm{tr}(X^TX)$.

Polar Decomposition

Given an arbitrary $n\times n$ matrix A that is not necessarily symmetric, let us connect the frame of columns to the springs, as in Figure 1, and then release. After undergoing a rotation $R\in SO(n)$, the frame will settle (assuming some damping) to the orientation of least potential energy; this new frame corresponds to a symmetric matrix S. In short, $S=RA$, i.e., $A=R^{-1}S$, which almost amounts to the polar decomposition of A. The "almost" is due to the fact that S need not be positive definite, as Figure 1b illustrates; one must first compose A with an extra reflection if $\det A<0$, and then carry out the above operation.

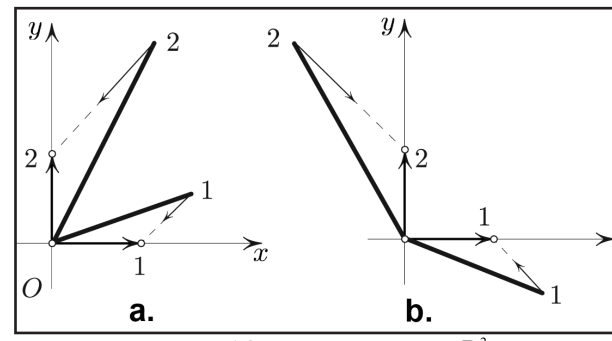

Figure 2. *Illustration of Sylvester's criterion in* $\mathbb{R}^2$.

Sylvester's Criterion

Sylvester's criterion is a necessary and sufficient condition for the positivity of a symmetric matrix that requires all principal minors to be positive. Minimality of the Toeplitz norm $\|A-I\|^2$ for positive definite 2×2 matrices makes Sylvester's criterion visually transparent. For example, if the Toeplitz norm is minimal for a positively oriented frame, only two cases of the equilibria shown in Figure 2 can occur; $a_{11}>0$ in both of these cases. Finding a purely visual proof of Sylvester's criterion for $n=3$ in a similar spirit is left as a challenge.

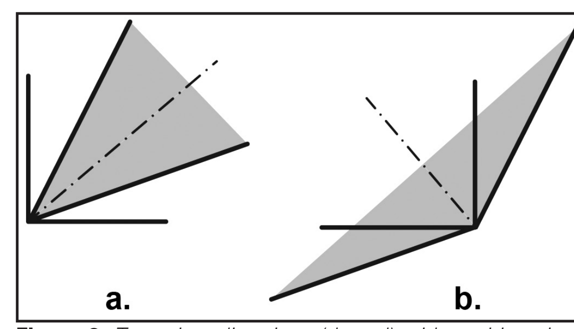

Figure 3. *Two eigendirections (dotted) with positive eigenvalues for the matrix in Figure 2a. Shaded sectors are* $A(Q2)$ (**3a**) *and* $A(Q1)$ (**3b**).

The figures in this article were provided by the author.

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