Orthocenter, Archimedean Style

MATHEMATICAL CURIOSITIES

By Mark Levi

The concurrency of altitudes in a triangle (at the point called the *orthocenter*) has been known since the times of Euclid, if not before. A couple of millenia later, Euler, Gauss, and other mathematicians came up with different proofs (see [2]). Arnold [1] observed that the Jacobi identity

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}$$
$$+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$$

for the cross product of vectors in $\mathbb{R}^3$ implies the concurrency of altitudes.

Writing the previous issue's column, "A Perspective on Altitudes,"[1] led me to the following physically-motivated proof of the concurrency of altitudes.

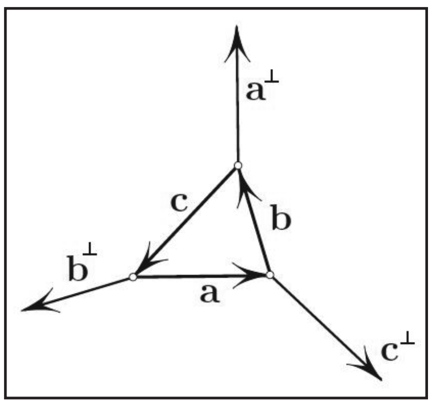

Figure 1. *The three forces balance; their lines are therefore concurrent.*

Our triangle is a rigid frame, free to slide within the plane. To each vertex of the triangle, let us apply the force perpendicular to the opposite side and of magnitude equal to that side's length (see Figure 1). I claim that *the triangle will remain in equilibrium*, i.e., the sum of these forces vanishes, as does the sum of their torques (relative to some pivot and hence to all pivots; the sum of torques is pivot-independent if the sum of forces is zero). Apart from being perhaps of independent interest, this equilibrium statement implies (and also follows from) the altitudes' concurrency.

Indeed, the contrary assumption—that the lines of forces are not concurrent, as in Figure 2—implies a nonzero torque relative to P, a contradiction.

It remains to prove that the forces and the torques in Figure 1 do indeed balance out. The total force

$$\mathbf{a}^\perp+\mathbf{b}^\perp+\mathbf{c}^\perp=\mathbf{0} \qquad (1)$$

since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. And the total torque vanishes according to Figure 3, where one of the vertices is chosen as a pivot. This completes the proof.

Thus, one could loosely say that the root cause of the altitudes' concurrency is the symmetry of the torque's magnitude under permutation of the lever and the force.

Incidentally, (1) is one of many corollaries of a perpetual motion machine's impossibility. Indeed, let our triangle be surrounded by the two-dimensional gas of pressure $p=1$ (units of force per unit of length). The sum in (1) is then the total force on the triangle, and must vanish since the alternative is a functioning perpetual motion machine.

The sum of torques of these forces vanishes as well, again by the perpetual motion argument. This also implies the concurrency of the midpoint perpendiculars — an alternative proof of the simple fact well known since antiquity.

I would not be surprised if someone in ancient times—Archimedes, perhaps—already came up with the above proof; we will probably never know. This suggests that if one waits long enough, an old idea becomes original.

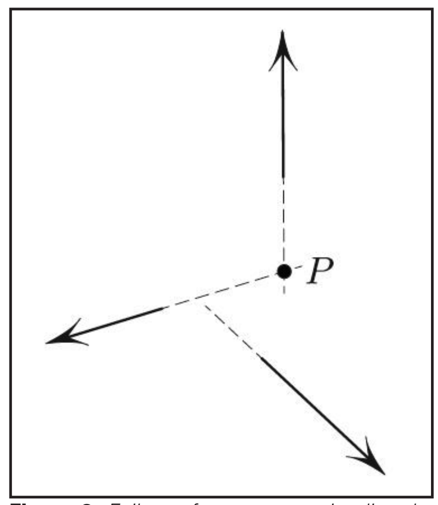

Figure 2. *Failure of concurrency implies the nonvanishing of the torque.*

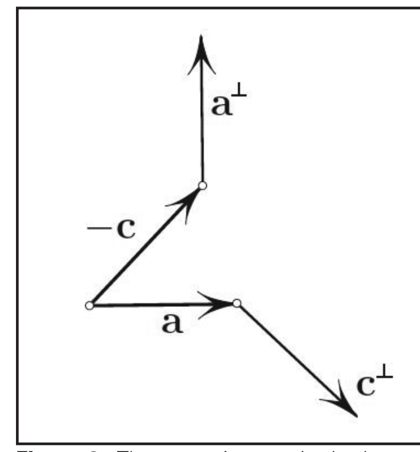

Figure 3. *The torque's magnitude does not change if the force and the lever are interchanged and rotated by* $\pi/2$.

The figures in this article were provided by the author.

References

[1] Arnold, V. (2005). Lobachevsky triangle altitude theorem as the Jacobi identity in the Lie algebra of quadratic forms on symplectic plane. *J. Geom. Phys., 53*(4), 421-427.

[2] Ivanov, N.V. (2011). Arnol'd, the Jacobi Identity, and Orthocenters. *Amer. Math. Mon., 118*(1), 2011, 41-65.

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[1] https://sinews.siam.org/Details-Page/a-perspective-on-altitudes