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A new randomness-generating mechanism in forced relaxation oscillations

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Abstract

This note points out that in a wide class of forced relaxation oscillators the hyperbolic behavior, which has up to now been known to exist only on small set, dominates in the Lebesgue sense in a wide class of relaxation oscillators. This note introduces the simplest physically realistic smooth system where the strange attractor is expected to exist for most (in the Lebesgue sense) parameter values. This phenomenon went unobserved for over half a century since the first work on forced relaxation oscillations by Cartwright, Littlewood and Levinson. Copyright © 1998 Elsevier Science B.V.

1. Background

Forced relaxation oscillations arise naturally as models of nonlinear electric circuits with positive feedback (among many other applications), and have been studied since the early work of van der Pol and van der Mark [15]. "Deterministic randomness" has been exhibited in this system by Cartwright, Littlewood and Levinson in the late 1940s, well before the discovery of the Lorentz attractor. The latter, however, shows the chaotic behavior in experiments, while the former does not. In fact, it has been shown earlier that the set of "chaotic" solutions in forced van der Pol-type systems occupies zero measure set in phase space, and thus is physically unobservable. Despite the great progress in our understanding of low-dimensional dynamics there had been no known differential equations for which the existence of chaos for the set of initial conditions of *positive measure* has been proven. It should be mentioned in this connection that chaotic attractors of positive measure should be expected on the basis of recent work by Benedicks and Carleson [1] and the (yet unpublished) work by Jakobson and Newhouse [8].

2. The main result

It has been known for over a decade that van der Pol-type relaxation oscillators with forcing exhibit chaotic behavior with zero probability, i.e. that almost all initial conditions give rise to simple periodic behavior, while only exceptional (unobservable) solutions are "chaotic". This note describes a previously unnoticed basic mechanism by which forced relaxation oscillations can produce chaotic behavior with *"positive probability"* – i.e. for a set of initial conditions of *positive* measure, and in fact for the majority of solutions. This phenomenon is sketched in

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Section 6, with the necessary geometric background given in Sections 3–5.

It is somewhat surprising that this simple basic mechanism has been overlooked in the half century since the original work by Cartwright and Littlewood (this work culminates in Littlewood's paper [13]). This is a mechanism by which "chaos" can be generated by nonlinear electric circuits, where "chaos" is meant in the physical, i.e. experimentally observable sense, as opposed to the "topological chaos" which one obtains from horseshoe maps and which gives zero measure sets of "interesting" initial conditions. (The hyperbolic set in the zero-measure case is, nevertheless, physically significant in providing the "watershed", i.e. the basin boundary for coexisting attractors.)

3. Geometry of the phase flow

In all forced relaxation oscillators studied previously [6,10,12–14] the phase flow stretches only a minority of initial conditions, while almost all other solutions are attracted to stable periodic solutions. This prevalence of contraction over most of the phase space is responsible for the fact that the hyperbolic ("chaotic") sets of solutions in relaxation oscillators studied up to now have zero measure; this explains why deterministic randomness has not been observed in numerical and physical experiments on relaxation oscillators [3,9].

A typical forced relaxation oscillator is given by

$$\begin{aligned} \dot{x} &= \frac{1}{\epsilon}\Phi(x, y) \\ \dot{y} &= -\epsilon x + bp(t), \end{aligned} \tag{1}$$

where the forcing p is taken with zero average.

The two cases of singular curves $\Phi(x, y) = 0$, one for the van der Pol-type equations and the other for the case discussed here, are shown in Fig. 1. We describe the dynamics of the oscillator as a "movie". [1]

[1] For the "bare essence" heuristic description of the stretching phenomenon the reader may wish to skip to Section 6. It should be pointed out, however, that the description does not explain why most initial conditions stretch – for that one has to "watch" the "movie" described here.

There are three key properties of the flow that combine to give the interesting behavior.

Property 1. The fast horizontal flow to the left below the singular curve and to the right above it, leading to the compression towards the left and right branches and the repulsion from the middle branch of $\Phi(x, y) = 0$.

Property 2. The near-periodic vertical oscillations: integrating the second equation in (1) we obtain

$$y(t) = y(0) - \epsilon \int_0^t x(\tau)\,\mathrm{d}\tau + bP(t),$$

where $P(t) = \int_0^t p(\tau)\,\mathrm{d}\tau$ is periodic since p has zero average.

Property 3. The vertical shear $-\epsilon x$.

These effects are listed in the decreasing order of their magnitudes $\mathrm{O}(\epsilon^{-1})$, $\mathrm{O}(\epsilon^{0})$ and $\mathrm{O}(\epsilon)$, respectively; we show now how these effects determine the shape of the Poincaré map, as was done in [10]. To that end we start with a large disc D of initial conditions; by Property 1 it collapses onto the shape shown in Fig. 2; this happens in a short time (on the order of $\mathrm{O}(\epsilon \ln \epsilon)$), and the solutions will stay captured in that set for all time. We now trace the consequent deformation of this set under the flow.

By Property 2, the y-coordinates reach their maximum (roughly) when $P(t)$ is maximal; let us assume that it happens at half-period $t = T/2$ – this amounts to the assumption that $p(t) > 0$ for $0 < t < T/2$ and $p(t) < 0$ for $T/2 < t < T$. As t changes from $\mathrm{O}(\epsilon \ln \epsilon)$ to $t = T/2$, the points in the neighborhood of the left branch of the singular curve slide upwards along that branch; if they reach its end, [2] they fall onto the right branch. The image $\phi^{T/2}(D)$ is shown in Fig. 2; here ϕ^t is the t-advance map for Eq. (1), i.e. $\phi^t(z)$ denotes the solution whose initial condition is z at $t = 0$.

[2] With enough time to spare before the end of the half-period.

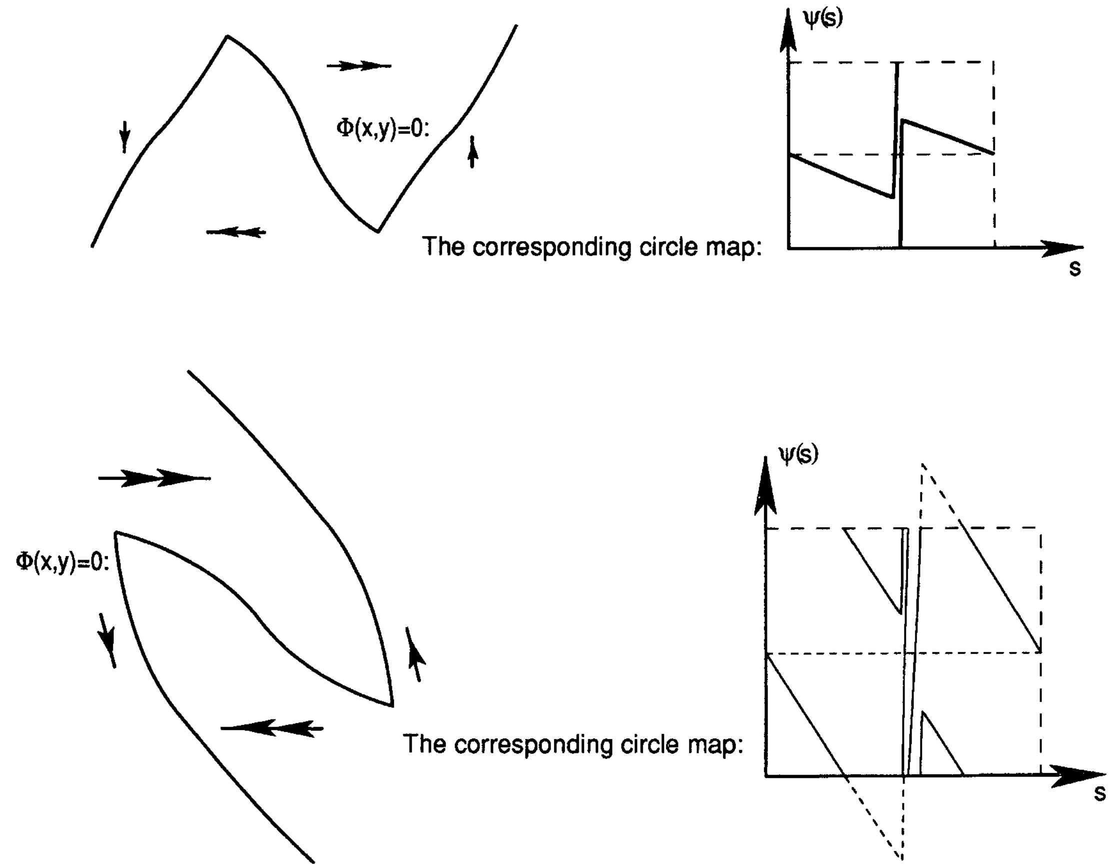

Fig. 1. The singular curve for the classical van der Pol case and for the case with chaotic attractor.

As t increases further from $T/2$ to NT with $N \approx 1/\epsilon$, the "ears" along the two stable branches retract gradually, due to the shear (Property 3), and the resulting picture is shown in Fig. 2. It should be noted in particular that the vertical dimension of $\phi^{NT}(D)$ is not $2m$, as one might expect, but $2m - b\bar{P} + o(\epsilon^0)$, Fig. 2, as we now explain. To estimate this vertical dimension we note that the vertical travel during $0 \leq t \leq T/2$ for each phase point is given by

$$y(T/2) - y(0) = b\int_0^{T/2} p(\tau)\,\mathrm{d}\tau - \epsilon\int_0^{T/2} x(\tau)\,\mathrm{d}\tau \equiv b\bar{P} + O(\epsilon).$$

Any portion of the set $\phi^t(D)$ that rises along the left branch above the level $y = m$ is swept to the right by the fast flow; similarly, anything that descends during $T/2 \leq t \leq T$ along the right branch below $y = -m$ is swept to the left. This process of vertical oscillations "shaves off" any portion of the set $\phi^t(D)$ which sticks out of the strip $-m < y < m$ of width $2m$. It follows that the approximate vertical width of the set $\phi^t(D)$ is $2m - b\bar{P}$ for sufficiently large t. In particular, for the dynamics to be nontrivial we assume $2m - b\bar{P} > 0$, i.e. $b < 2m/\bar{P}$.

Similiar arguments (cf. [10]) show that a properly chosen small disk D_{in} surrounding the unstable fixed point of ϕ^T expands, and thus there exists an annulus A which maps into itself under ϕ^T. The outer boundary of A contains $\phi^{nT}(D)$ for all n large enough, while the inner boundary of A is contained in $\phi^{nT}(D_{\text{in}})$. Proof of these statements can be adapted from [10] and we do not give them here. Since one ϕ^T-iteration contracts

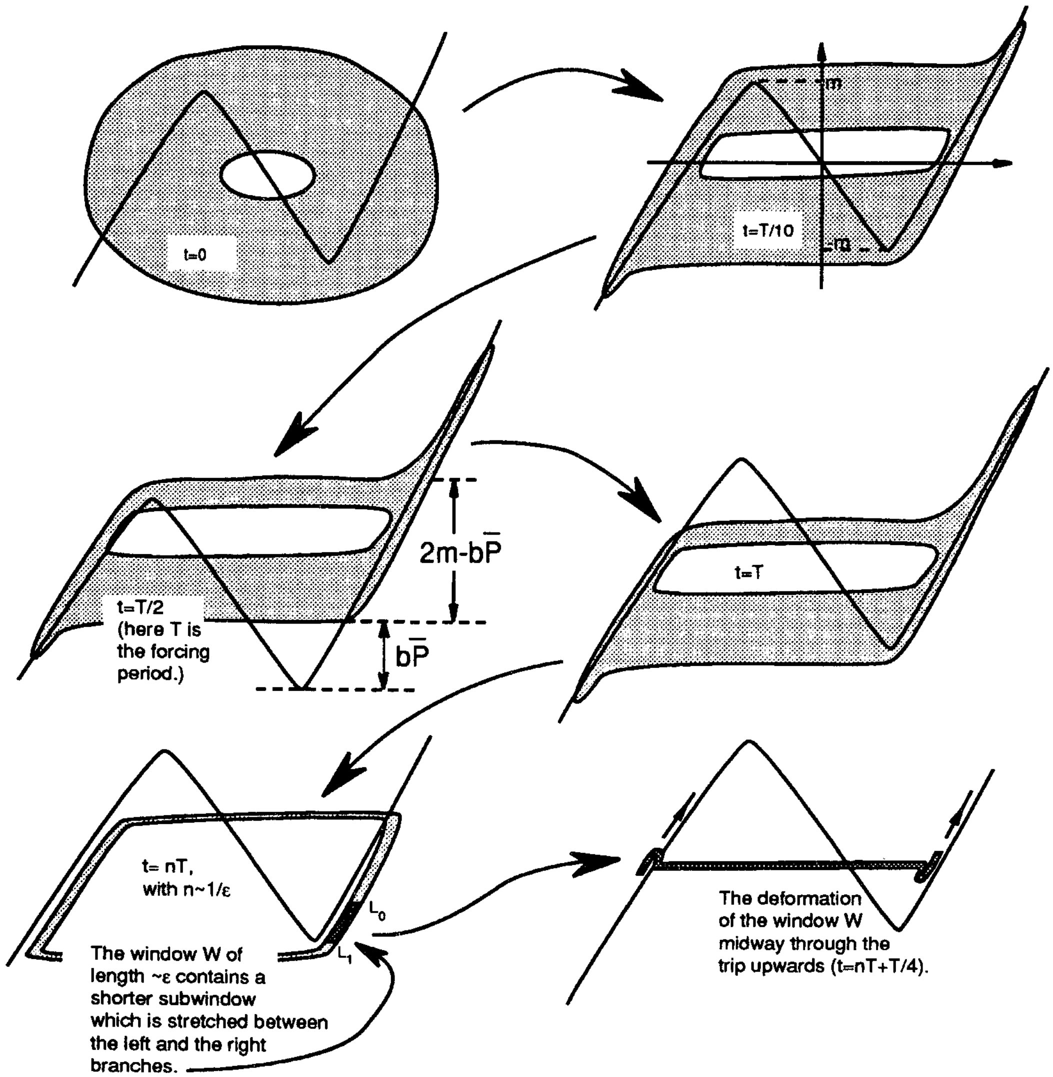

Fig. 2. Geometry of the phase flow. A "fat" initial annulus A becomes thin after $O(1/\epsilon)$ iterations.

towards the stable branch by the factor of $e^{-c/\epsilon}$, and since all points spend $O(1/\epsilon)$ iterations near a stable branch before jumping to the other stable branch, the annulus A can be constructed so that its thickness near the ends of stable branches is $O(e^{-c/\epsilon^2})$.

4. The window map and the circle map

We define the "window" W in the phase plane, Fig. 2, by choosing a horizontal segment L_1 connecting the inner and the outer boundaries of the annulus as shown in figure, letting $L_0 = \phi^{-T} L_1$ and letting W be the part of the annulus A which lies between L_0 and L_1. By this defintion, every point in the phase plane, with the exception of the unstable fixed point of ϕ^T visits W once and hence repeatedly. We define the window map $\Psi_W : W \to W$ as the first return map to W. In other words, Ψ_W is the Poincaré map of the flow on the extended phase space with $W \times \{t = 0 (\mathrm{mod}\ T)\}$ as the Poincaré section. Equivalently, $\Psi_W(z) = \phi^{jT}(z) \in W$, where $j = j(z) >$

0 is the smallest integer for which the last inclusion holds. Since j depends on z, the map is discontinuous. This discontinuity is removed, however, if we identify each point with its ϕ^T-iterate: indeed, the discontinuity manifests itself in jumps by one iterate, and the identification removes precisely such jumps. At the same time the identification turns W into an annulus; to summarize, we obtain a continuous map of the annulus W. The entire qualitative behavior of the differential equation (1) is captured by the window map $\Psi_W : W \to W$ (without risking a confusion we keep the notation W for the annulus). In particular, periodic solutions of the differential equation correspond to periodic points of Ψ_W; stability of the former coincides with stability of the latter, and so on. The only information missing in the annulus mapping, namely, the length of the period of solutions, can be easily recovered from simple additional considerations [10].

4.1. Rescaling and the circle mapping

Since the length of W is $\mathrm{O}(\epsilon)$ while the thickness is $\mathrm{e}^{-c/\epsilon^2}$ it is convenient to introduce a rescaling transformation $S : W \to W_0 = [0, 1] \times [0, \mathrm{e}^{-1/\epsilon^2}]$ which normalizes the window map to the form $\Psi_N = S \circ \Psi_W \circ S^{-1}$. We also define the associated projected circle map $\psi_N : S^1 \to S^1$ by $\psi_N(s) = \pi_1 \circ \Psi_N(0, s)$, where π_1 is the projection onto the first coordinate axis.

4.2. Factorization by symmetry

Symmetry properties of $\Phi(x, y)$ and of $p(t)$ translate into a nice symmetry property of the window map: it turns out that Ψ_N can be expressed as a second iterate $\Psi_N = \Psi \circ \Psi$ of a simpler annulus mapping Ψ whose projected circle map is shown in Fig. 1. The proof of this statement can be found in [10] and in [11]. We only mention here that Ψ is the antipodal return half-period map, i.e. Ψ is defined by rescaling as mentioned above the map $z \mapsto -\phi^{nT+T/2}z \in W$, where $n = n(z_0) > 0$ is the smallest integer for which the last inclusion holds. For the projected circle map we have $\psi_N = \psi \circ \psi$; a possible graph of ϕ is sketched in Fig. 1. To summarize these preparatory remarks, the study of the relaxation oscillator reduces to the study of an annulus map $\Psi : W_0 \to W_0$ and of the projected circle mapping ψ. The advantage of such reduction lies in the greater simplicity of the mapping $\Psi = \sqrt{\Psi_N}$ in comparison to the map Ψ_N and especially in comparison with ϕ^T.

5. Behavior of the window mapping

The gist of the main result of the paper is contained in the following theorem, according to which the projected circle mapping and hence the Poincaré map itself is stretching on most of its domain. We recall that $\Psi : W_0 \to W_0$ is the normalized antipodal half-period return mapping associated with the system (1) as described above, and $\psi : \mathbf{R}(\mathrm{mod}\ 1) \to \mathbf{R}(\mathrm{mod}\ 1)$ is the corresponding projected circle map.

Theorem. If the vectorfield (1) satisfies conditions (A)–(C) below, and if $\kappa < b < 2m/\bar{P} - \kappa$ for some $\kappa > 0$, then for all sufficiently small ϵ the Poincaré map ϕ^T of (1) is stretching on most of its domain. More precisely, for the circle map ψ there exists a constant $\alpha > 1$ independent of ϵ and an interval I_ϵ on the circle $\mathbf{R}(\mathrm{mod}\ 1)$, with $|I_\epsilon| < \sqrt{\epsilon}$, such that

$$|\psi'(s)| \geq \alpha > 1 \quad \text{for all } s \text{ outside } I_\epsilon. \tag{2}$$

The 2D window map Ψ has a similar behavior: there exists a family of vertical and horizontal sectors of tangent vectors:

$$H_\mu = \{(\xi, \eta): |\eta| < \mu|\xi|\},$$
$$V_\mu = \{(\xi, \eta): |\xi| < \mu^{-1}|\eta|\}, \quad \mu = \mathrm{O}(\epsilon),$$
$$\mu^{-1} = \mathrm{O}(\epsilon^{-1})$$

such that at all $z = (x, y) \in W_0$ with $x \notin I_\epsilon$ the horizontal sector at z maps into the horizontal sector at Ψz under $\mathrm{d}\Psi$, similarly for vertical sectors:

$$(\mathrm{d}\Psi)H_\mu \subset H_\mu, \qquad (\mathrm{d}\Psi^{-1})V_\mu \subset V_\mu,$$

with proper stretching: for $\zeta_i = (\xi_i, \eta_i) = \mathrm{d}\Psi_z^i(\xi_0, \eta_0)$, $i = \pm 1$, we have

$\zeta_0 \in H_\mu$ implies $\xi_1 > \alpha\xi_0$

and

$$\eta_0 \in V_\mu \text{ implies } \eta_{-1} > K_\epsilon \eta_0,$$
$$K_\epsilon = \mathrm{O}(\mathrm{e}^{c/\epsilon^2}), \quad c > 0. \tag{3}$$

Inside the interval I_ϵ the following holds: there is a subinterval $I'_\epsilon \subset I_\epsilon$ such that the same estimates as above hold for I'_ϵ with the stretching estimate (3) strengthened to

$$\zeta_0 \in H_\mu \text{ implies } \xi_1 > \beta_\epsilon \xi_0,$$
$$\beta_\epsilon \geq \mathrm{e}^{c/\epsilon} \gg 1, \quad c > 0. \tag{4}$$

Conditions

(A) The singular curve $\Phi(x, y) = 0$ consists of two stable branches and of one unstable branch as shown in Fig. 1, all with negative slopes:

$$\frac{\Phi_x}{\Phi_y} > 0 \quad \text{for all } (x, y) \text{ with } x \neq \pm 1.$$

Morever, let $(-1, m)$ and $(1, -m)$ be the coordinates of the two points where the unstable branch meets the stable branches (there is no loss of generality in assuming the x-coordinates to be ± 1.) An extra requirement is that $\Phi(x^*, m) = 0$ has a *positive* root $x^* > 0$. Equivalently, this condition amounts to the assumption that the part of the right stable branch inside the strip $|y| < m$ lies entirely in the right half-plane where the drift $-\epsilon x$ is downwards; similarly, for the left stable branch.

(B) $\left|\frac{\Phi_x}{\Phi_y}\right| \geq c > 0$ and $|\Phi_x(x, y)| \geq c > 0$ for all (x, y); in particular, all slopes have positive lower bounds.

(C) The forcing term $p(t)$ is piecewise continuous, periodic of period $T > 0$, satisfies $p(t) \geq c > 0$ for all t outside a sufficiently small interval, and $p(t) > 0$ for $0 < t < T/2$ and $p(t) < 0$ for $T/2 < t < T$. We assume in addition that $p(t)$ is odd and that $p(t + T/2)$ is odd.

It has not been proven that the map Ψ satisfying the conclusions of the above theorem, together with some additional assumptions, posesses a transitive attractor with positive Lyapunov exponents, for a set of b-values of positive Lebesgue measure. The experimental significance of the above theorem, however, does not depend on these rather technical and difficult points: for practical purposes, the behavior in the situation described here will look random for most parameter values, whether or not the attractor is chaotic in a rigorous sense.

6. A heuristic explanation of the stretching phenomenon

The key to the explanation lies in understanding the flow near the stable branches of the singular curve $\Phi(x, y) = 0$. A "typical" solution $z(t) = (x(t), y(t))$ is confined to an $\mathrm{O}(\epsilon)$-tubular neighborhood of a stable branch for all times except for the relatively short transition intervals.[3] Consider the linearized equations around such a solution:

$$\dot{\xi} = \frac{1}{\epsilon}(\Phi_x(x, y)\xi + \Phi_y(x, y)\eta),$$
$$\dot{\eta} = -\epsilon\xi. \tag{5}$$

This system inherits the relaxation character from (1), and after a short settling relaxation time[4] we obtain

$$\Phi_x \xi + \Phi_y \eta = \mathrm{O}(\epsilon),$$

which gives

$$\xi = -\frac{\Phi_y}{\Phi_x}\eta + \mathrm{O}(\epsilon).$$

Substitution of this relation into the second equation of (5) gives, after dropping the $\mathrm{O}(\epsilon^2)$-terms, the estimate

$$\dot{\eta} = \epsilon k(x, y)\eta, \qquad k = \frac{\Phi_y}{\Phi_x}.$$

[3] Here "typical" refers to the majority, in the sense of Lebesque measure, of trajectories which do not follow the unstable branch. These latter trajectories which do follow the unstable branch are responsible for the short interval I_ϵ where the circle map is very strongly expanding.

[4] In this heuristic discussion we do not specify the precise meaning of shortness of time intervals, etc.: the details will appear elsewhere.

It should be noted that $k = -1/(\text{slope of } \{\Phi(z) = 0\})$, and thus $k > 0$ for as long as $z(t)$ is near the stable branch. This explains the stretching phenomenon:

$$\frac{\eta(T)}{\eta(0)} = e^{\epsilon \int_0^T k(z(t))\,dt} > 1 + c\epsilon,$$

$c > 0$ independent of ϵ,

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with c independent of ϵ. Since the map Ψ involves $O(1/\epsilon)$ iterations of ϕ^T, this implies the stretching property of the circle map, as stated in the theorem. Detailed proofs will appear elsewhere.

Among the open questions there remains a difficult problem of proving the same properties for the case when the singular curve is smooth.

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