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BOUNDEDNESS OF SOLUTIONS FOR QUASIPERIODIC POTENTIALS*

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Abstract. In this paper conservative systems are studied describing the motion of a particle on the line in the field of a potential force with additional quasiperiodic time dependence.

It is shown that superquadratic growth of the potential at infinity results in the near-integrability of the Hamiltonian system in question (for a large class of potentials), despite the fact that no smallness assumptions are made on the quasiperiodic dependence of the potential on time. As a consequence all the solutions of such systems are bounded for all time. Some specific examples are given, together with a counterexample which shows that, without the quasiperiodicity assumption, the boundedness breaks down.

Key words. quasiperiodicity, stability, action-angle variables, normal term, KAM theory

AMS subject classifications. 34, 58

1. Introduction and results.

We shall study the boundedness of all solutions of time-dependent equations having the form

$$\ddot{x} + V_x(x, \omega t) = 0, \quad x \in \mathbf{R}. \tag{1}$$

This equation is the simplest yet highly nontrivial model of conservative systems such as charged particles in periodic fields. Setting $\dot{x} = y$, these equations can be rewritten in Hamiltonian form with Hamiltonian functions

$$H(x, y, t) = \frac{1}{2}y^2 + V(x, \omega t) \tag{2}$$

on the extended phase space $(x, y, t) \in \mathbf{R}^3$.

A distinguished class of equations describe forced pendulum-like systems in which the potential is assumed to be periodic in x such that V is a function on $S^1 \times \mathbf{R}$. If, in addition, the time dependence is periodic or even quasiperiodic, then it turns out that every solution $x(t)$ of (1) is bounded in the phase space $S^1 \times \mathbf{R}$, i.e., $\sup\{|\dot{x}(t)|, t \in \mathbf{R}\} < \infty$, provided that only the potential V is sufficiently smooth and, in the quasiperiodic case, the frequencies meet a Diophantine condition. The proof of this phenomenon is based on the observation that such systems are near so-called integrable systems in the region of the phase space $S^1 \times \mathbf{R}$ in which y is sufficiently large. For proofs we refer to Levi [12], Moser [13], and Chierchia and Zehnder [14]. If, however, the smoothness requirements are not met, for example, if V merely belongs to the class C^2, then unbounded solutions may be expected. Also, if the above restrictions on the time dependence are dropped, unbounded solutions are likely to occur even for smooth and bounded potentials V.

If the periodicity requirement in x is dropped, then the configuration space is no longer S^1 but $\mathbf{R}^1$, and the question of boundedness of all solutions for (1) is much more subtle. It is related to the asymptotic behavior of the nonlinearity in x, the

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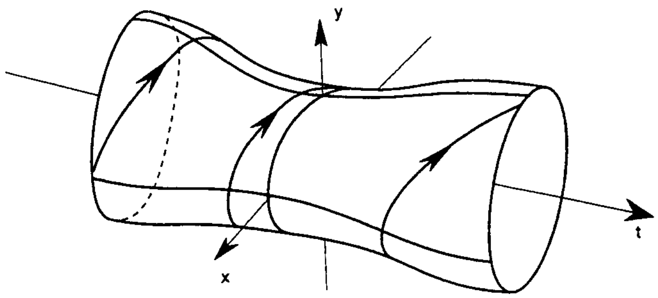

FIG. 1.1.

smoothness of V, and also the nature of the time dependence. For example, if

$$H(x,y,t) = \frac{1}{2}y^2 + \frac{1}{\sqrt{x^2 + r(t)}}$$

for a time periodic and positive function $r(t)$, as in the case of the so-called restricted three-body problem, then the solutions with initial conditions $\dot{x}(0) = y(0)$ sufficiently large are clearly not bounded. Here the level lines of $H(x, y, t) = E$ for frozen t are not closed curves if E is large. In contrast, in the example

$$\ddot{x} + a(t)x^3 + b(t)x^2 + c(t)x = p(t) \tag{3}$$

level lines for E large are closed curves; see Fig. 1.1. However, the energy is not conserved in time and might increase, forcing a solution to be unbounded in the phase space $\mathbf{R}^2$. For this class of examples the subtle question of boundedness of solutions was already raised by Littlewood, who constructed examples [2] admitting unbounded solutions assuming a periodic but discontinuous forcing $p(t)$; see also Levi [3] and Long [4]. Recently Zharnitsky [30] succeeded in constructing such an example with $p(t)$ discontinuous. In 1976 Morris [5] succeeded in proving that, for a continuous time periodic forcing, all the solutions of

$$\ddot{x} + x^3 = p(t)$$

are bounded in $\mathbf{R}^2$. For more recent results in the time periodic case we refer to [6]–[9], [16]. We also mention the related problem of Ulam and Fermi's "ping-pong," consisting of a particle bouncing between a wall and a periodically moving "paddle" parallel to the wall, undergoing perfectly elastic collisions with both; a basic physical question is whether the energy of a particle stays bounded for all time in such a periodically varying system. It should be mentioned that this problem is a limiting case of the aforementioned problems where the walls of the potential well become infinitely steep. The affirmative answer to the last question for sufficiently smooth periodic motions of the wall has been given by Moser (in an unpublished, private communication), Douady [27], and in [6]. In what follows we shall assume that the time dependence in (1) is quasiperiodic with frequencies $\omega = (\omega_1, \ldots, \omega_N) \in \mathbf{R}^N$; i.e., we assume

$$V(x, \omega t) = V(x, \omega_1 t, \ldots, \omega_N t), \tag{4}$$

where $V(x, \xi_1, \ldots, \xi_N)$ is assumed to be periodic of period 1 in all the variables $\xi_1, \ldots, \xi_N$. Moreover, we shall assume that the frequencies ω are not only rationally independent, but meet the Diophantine conditions

$$|\langle \omega, j \rangle| \geq \gamma |j|^{-\tau} \quad \text{for all } j \in \mathbf{Z}^N \backslash \{0\} \tag{5}$$

with two constants $\tau > N$ and $\gamma > 0$. The brackets on the left-hand side denote the scalar product.

The system considered first is of the form

$$V(x, \omega t) = \sum_{j=1}^{2n+2} a_j(\omega t) x^j, \tag{6}$$

$n \geq 1$, where all the coefficients are quasiperiodic functions in time with the same frequencies $\omega \in \mathbf{R}^N$ and, in addition, the leading coefficient $a = a_{2n}$ is positive:

$$a(\omega t) \geq \min_{\xi \in T^N} a(\xi) > 0. \tag{7}$$

1.1. Stability and invariant tori for polynomial potentials.

THEOREM 1. *Let the polynomial potential $V(x, \omega t)$ satisfy* (6) *and* (7) *together with the Diophantine conditions* (5), *and assume that $a_j \in C^k(T^N)$ for $k > 4\tau + 6$ and $0 \leq j \leq 2n + 2$. Then all solutions $x(t)$ of $\ddot{x} + V_x(x, \omega t) = 0$ are bounded, i.e.,*

$$\sup_{t \in \mathbf{R}} (x(t)^2 + \dot{x}(t)^2) < \infty.$$

Note that the smoothness requirement depends only on the number of the underlying frequencies ω and not on the degree of the polynomial in x. Already for the time periodic case this statement is not quite obvious. It has only recently been proved by Laederich and Levi in [6], improving and simplifying an earlier result of Dieckerhoff and Zehnder [7]. We should point out that all the boundedness proofs in the time periodic case use Moser's twist theorem [19] and its regularity improvements [10] and [11] in a crucial way.

In order to describe the idea of the proof in the quasiperiodic case, we write the equation as a system in the phase space $\mathbf{R}^3$,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -V_x(x, t), \\ \dot{t} &= 1, \end{aligned} \tag{8}$$

and abbreviate the vector field in $\mathbf{R}^3$ on the right-hand side by X. For $C > 0$ we denote by A_C the region $A_C = \{(x, y, t) | x^2 + y^2 > C\}$ in $\mathbf{R}^3$. For every $C > 0$ we shall construct an embedded cylinder $w : \mathbf{R} \times S^1 \to \mathbf{R}^3$ contained in A_C,

$$w : (t, s) \mapsto (u(t, s), v(t, s), t), \tag{9}$$

satisfying

$$C < \inf_{\mathbf{R} \times S^1} (u^2 + v^2) < \sup_{\mathbf{R} \times S^1} (u^2 + v^2) < \infty$$

and which is tangential to the vector field X in $\mathbf{R}^3$, so that it is invariant under the flow of X. Now if a solution $(x(t), y(t), t)$ of (8) satisfies $x(t^*)^2 + y(t^*)^2 \leq C$ for some

$t^* \in \mathbf{R}$, then it follows from the invariance of the cylinder and the uniqueness of the solutions that this solution does exist for all times $t \in \mathbf{R}$ and satisfies, in addition, $(x(t)^2 + y(t)^2) \leq \sup_{\mathbf{R}\times S}(u^2 + v^2) < \infty$ for all $t \in \mathbf{R}$. Hence it is bounded.

The existence of these invariant surfaces in $\mathbf{R}^3$ will be concluded from the observation that the system (8) is in the region A_C near an integrable system, provided that C is sufficiently large, so that well-known small denominator perturbation techniques can be applied. These techniques require an excessive amount of smoothness; this is, of course, well known. The near integrability is, however, not obvious a priori and its proof is the main task. It will be apparent only after scaling the time t and the phase space variables and only after several coordinate changes, which transform the vector field into a suitable form.

The invariant surface found consists of quasiperiodic solutions, and we shall prove the following existence statement.

THEOREM 2. *The equation*

$$\ddot{x} + V_x(x, \omega t) = 0$$

with the potential V satisfying the assumptions of Theorem 1 *possesses uncountably many quasiperiodic solutions having* $1 + N$ *frequencies* $(\alpha, \omega) \in \mathbf{R}^{1+N}$ *and satisfying the Diophantine conditions*

$$|\alpha k + \langle \omega, j \rangle| \geq \gamma(|k| + |j|)^{-\tau}$$

for all $(k, j) \in \mathbf{Z}^{1+N}\backslash\{0\}$ *with the same constants* $\gamma > 0$ *and* $\tau > N$ *as in* (5).

Indeed, as expected, the dominant part of the phase space for $x^2 + \dot{x}^2$ large is covered by quasiperiodic solutions. The worst possible failure of the Diophantine condition (5) corresponds to all frequencies ω being rational multiples of one of them; in this case V is time periodic and the aforementioned results for the periodic case apply, showing boundedness under appropriate assumptions.

The intermediate (Liouville) case between the periodic one and the Diophantine one is less clear. It seems highly likely that when ω is a Liouville vector, where (5) fails for infinitely many j, an arbitrarily small change in V would destroy an invariant surface with fixed frequencies if there is one; see Mather [21]. On the other hand, it is less clear whether all such tori can be destroyed at once, or perhaps the destruction of one torus could lead to the creation of another one.

1.2. General potentials. So far we have considered a rather restricted class of potentials. It turns out that the ideas of the proof of the aforementioned Theorems can be applied to a more general class of quasiperiodic potentials introduced recently in [15] for the time periodic case.

THEOREM 3. *If ω satisfies the Diophantine conditions* (5) *then there exist positive constants $a, b, \mu_0(b)$, and $\gamma(b)$ such that the conclusions of Theorems* 1 *and* 2 *hold for the equation*

$$\ddot{x} + V_x(x, \omega t) \;=\; 0$$

provided $V(x, \xi)$ with $x \in \mathbf{R}$ and $\xi \in T^N$ belongs to $C^d, d = 4\tau + 7 + \gamma(b)$ and, moreover, satisfies the following conditions:

(i)

$$V(x, \xi) \to \infty \;\; as \;\; |x| \to \infty$$

uniformly in $\xi \in T^N$.

(ii) *In the notation*

$$W := \frac{V}{V_x} \text{ and } U := \frac{V_\xi}{V_x},$$

the following estimates hold for all $(x, \xi) \in \mathbf{R} \times \mathbf{R}^N$ *and* $|x|$ *large:*

$$\text{(iia)} \qquad -\frac{1}{2} + a < \partial_x W(x,\xi) < \frac{1}{2} - b, \quad b < 1 - a,$$

$$\text{(iib)} \qquad |\partial_x^k \partial_\xi^\alpha V(x,\xi)| \le C_{k\alpha} |x|^{-k} |V(x,\xi)|^{1+\mu} \text{ for some } 0 \le \mu \le \mu_0(b)$$

and

$$|\partial_x^k \partial_\xi^\alpha U|, |\partial_x^k \partial_\xi^\alpha W| \le C_{k\alpha} |x|^{1-k}$$

for all $|k| + |\alpha| \le 4\tau + 6 + \gamma(b)$.

This theorem will be proved in §4.

Examples. The above conditions (i) and (ii) with $\mu = 0$ hold for polynomial (in x) potentials. With $\mu \neq 0$ the class of potentials widens considerably to include exponential growth, oscillatory growth, and much more. The simplest example is $V_1(x,t) = p(\omega t) \cosh x$; it satisfies all the conditions if $p > 0$ is smooth enough.

A more difficult potential

$$V_2(x,t) = p(\omega t)(\cosh x + q(x)),$$

where q is any polynomial, satisfies conditions (i) and (ii) as well, again provided $p : T^n \to \mathbb{R}$ is smooth enough. The polynomial $q(x)$ in the above example can be replaced by, say, $\cos x$ or a polynomial in x and $\cos x$ without violating the conditions of Theorem 3:

$$V_3(x,t) = p(\omega t)(\cosh x + q_1(x, \cos x)).$$

Yet another example is

$$V_4(x,t) = \cosh\left[(3 + \cos t + \cos\sqrt{2}t)(x + \sin(1 + x^2)^\nu)\right]$$

with $\nu > 0$ sufficiently (specifically) small.

The list can be continued indefinitely.

It should be emphasized that analogous stability results cannot be expected in higher dimensions. Consider, for example, a time-dependent Hamiltonian system defined by the Hamiltonian function

$$H(x,y,\omega,t) = \frac{1}{2}|y|^2 + V(x,\omega t)$$

on $(x,y) \in T^n \times \mathbb{R}^n$ for which the energy is not conserved. Here one also finds an abundance of quasiperiodic solutions in the region of the phase space where $|y|$ is large, in which the system can be considered as a system near an integrable one, provided V is sufficiently smooth; see, e.g., [14]. However, if $n > 1$ then the existence of these solutions does not lead to bounds for all solutions of the system. But there are bounds for all solutions not over an infinite interval of time but over an exponentially large interval of time, provided the potential is not only smooth but real analytic. This well-known phenomenon has been discovered by Nekhoroshev [25]. As an illustration we mention the effective bounds for the above example. Assume $V(t,x)$ with $(t,x) \in \mathbb{R}^n \times T^n$ has a holomorphic extension to an imaginary strip $|\operatorname{Im} t| \leq \sigma$ and $|\operatorname{Im} t| \leq \sigma$ for some positive number σ. Then there are positive constants T^* and R^* depending on V, σ, and the dimension n such that, for every $\rho \geq R^*$ and every solution $(x(t), y(t))$ of the Hamiltonian equation, we have

$$|y(t) - y(0)| \leq \rho$$

for all t in the interval

$$|t| \leq T^* \exp\left(\frac{\rho}{R^*}\right)^{\alpha}.$$

The proof of these estimates with explicit constants is based on Nekhoroshev's ideas, and we refer to [26] (with $\alpha = 2/(n^2+n)$) and [28] and [29] for the recently improved estimate with $\alpha = \frac{1}{2n}$. This is merely a special example of an exponential stability result which replaces the stronger stability results of Theorem 3 for systems in higher dimensions.

1.3. Unbounded solutions with nonrecurrent forcing. We shall show that, as soon as the quasiperiodicity requirement of the time dependence is removed, even the "nicest" equations can have unbounded solutions for forcing terms which are smooth, small, and tend to zero as the time goes to infinity.

THEOREM 4. *Given any $\varepsilon > 0$ and any $r \in \mathbf{N}$, there exists a function $p \in C^{\infty}(\mathbf{R})$ satisfying*

$$\|p\|_{C^r(\mathbf{R})} \leq \varepsilon \text{ and } \lim_{t\to\infty} D^j p(t) = 0 \text{ for } 0 \leq j \leq r-1, \tag{10}$$

such that the equation

$$\ddot{x} + x^3 = p(t) \tag{11}$$

possesses an unbounded solution $y(t)$. Moreover, the rate of decay in (10) *is given by*

$$\sup_{t>0} t^{\frac{2(r-j)}{2r+3}} |D^j p(t)| < \infty \tag{12}$$

for $0 \leq j \leq r-1$, and the rate of growth of the unbounded solution $y(t)$ is given by

$$\frac{1}{C} t^{\frac{4}{2r+3}} \leq \frac{1}{2}\dot{y}(t)^2 + \frac{1}{4}y(t)^4 \leq C t^{\frac{4}{2r+3}} \tag{13}$$

for $t \geq 1$ with a positive constant C depending on ε.

It should be pointed out that the first part of the theorem holds true for every equation $\ddot{x} + V_x(x) = p(t)$ provided that $\frac{1}{x}V_x(x) \to \infty$ as $|x| \to \infty$; i.e., an unbounded solution can be produced with a forcing satisfying (10). This will follow from the proof. As for another, more subtle phenomenon we recall that Coffman and Ullrich [22] constructed a positive and continuous function $p(t)$ close to a constant, which, however, is not of bounded variation near a point t^*, such that the equation $\ddot{x} + p(t)x^3 = 0$ has a solution which is unbounded on the finite interval $0 \leq t < t^*$.

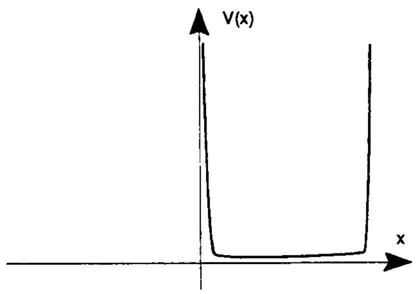

FIG. 2.1.

2. A "squash player's" potential and some open problems. Let $M > 0$ be a large integer and consider equation (1) with the special potential

$$V(x,\omega t) = (x-1)^{2M} + \sum_{i=1}^{n} (x/p_i(\omega_i t))^{2M},$$

where $0 < p_i(\tau) < 1$ are periodic functions of period 1. Since M is large, V has two steep "walls," one near $x = 0$ and the other near $x = \min_{1\le i\le n}\{p_j(\omega_i t)\}$; see Fig. 2.1. One may think of n squash players each moving his racket periodically, holding it at the distance $p_i(\omega_i t)$ from the wall $x = 0$. The player whose racket is in front, i.e., closest to the wall at a given moment, "gets to hit the ball." It would make sense to assume that $\min_\tau p_i(\tau) < \max_\tau p_j(\tau)$ for all $i, j = 1, \ldots, n$, so that everyone gets a chance to hit. Now Theorem 1 shows that as long as the frequency vector ω is Diophantine and $p_i \in C^k(S^1)$ with $k > 4\tau + 6$, the game will proceed without an escalation, i.e., both the speed and the position of any possible motion will stay bounded for all $t \in \mathbf{R}$. Of course, no explicit estimate on that bound is given and no estimate is given on how deep the potential wall is penetrated.

Open problems. 1. Modifying the "squash" example by making the walls rigid, we obtain a problem not covered by Theorems 1 or 2. In fact, it is doubtful that the result still holds for such a modification since the smoothness of the "potential" is lost in taking the rigid limit.

2. As for a different modification, one could consider the Ulam–Fermi "ping-pong" problem consisting of a particle bouncing elastically between two parallel walls with the walls undergoing a quasiperiodic motion. The problem is to prove that the velocity of every motion is bounded for all time provided the motion of the walls is smooth enough and the Diophantine conditions hold by establishing the existence of invariant cylinders in the extended phase space of the system.

3. Proof of Theorems 1 and 2. We first transform the equation

$$\ddot{x} + V_x(x,\omega t) = 0, \quad x \in \mathbf{R} \tag{14}$$

with V satisfying assumptions (4)–(7) into a suitable form. We proceed in several steps.

3.1. The rescaling into a slow system. As in [6] we first rescale the time variable t and, at the same time, the space variable x setting for small $\delta > 0$,

$$u = \delta x, \; t = \varepsilon s, \quad \text{where } \varepsilon = \delta^n. \tag{15}$$

If $x(t)$ is a solution of (14), then

$$u(s) := \delta x(\varepsilon s)$$

is a solution of the equation

$$\frac{d^2}{ds^2}u + \varepsilon^2 \delta V_x\left(\frac{u}{\delta}, \varepsilon\omega s\right) = 0, \quad u \in \mathbf{R}. \tag{16}$$

In view of the assumptions on V we are led to the equivalent differential equation

$$\frac{d^2}{ds^2}u + W_u(u, \varepsilon\omega s, \varepsilon) = 0, \tag{17}$$

where $W(u, \xi, \varepsilon)$, $\xi \in T^N$ is given by

$$W(u, \xi, \varepsilon) = a(\xi)u^{2n+2} + \varepsilon^\alpha \sum_{j=1}^{2n+1} a_j(\xi)\varepsilon^{\alpha(2n+1-j)}u^j \quad \text{with } \alpha = \frac{1}{n}$$

and $a \equiv a_{2n+2}$; moreover, $a_j \in C^k(T^N)$. Now, returning to the old notation by replacing u by x and s by t we therefore arrive at the Hamiltonian system

$$H(x, y, \varepsilon\omega t, \varepsilon) = \frac{1}{2}y^2 + W(x, \varepsilon\omega t, \varepsilon) \tag{18}$$

in the extended phase space $(x, y, t) \in \mathbf{R}^3$. Our aim is to construct, for every $\varepsilon > 0$, an invariant cylinder for (18) contained in $(x, y, t) \in A \times \mathbf{R}$ having the time axis in its interior, where A is a fixed and bounded annular region in $\mathbf{R}^2$ around the origin.

3.2. The action-angle variables. At first we consider the time-independent Hamiltonian system in $(x, y) \in \mathbf{R}^2$ given by

$$H(x, y, \xi, \varepsilon) = \frac{1}{2}y^2 + W(x, \xi, \varepsilon), \tag{19}$$

which depends on the parameters $\xi \in \mathbf{R}^N$ and $\varepsilon > 0$. The dependence on each ξ_i is periodic with each period 1. If ε is sufficiently small one can introduce in an annulus-like domain in the (x, y)-plane so-called action and angle variables $(\varphi, I) \mapsto (x, y)$, using a generating function $S(x, I) \equiv S(x, I, \xi, \varepsilon)$ depending periodically on ξ by the formula

$$\begin{aligned} y &= S_x(x, I), \\ \varphi &= S_I(x, I). \end{aligned} \tag{20}$$

As usual [23], the action variable I is defined as the area of the level curve γ in $\mathbf{R}^2$ defined by $H(x, y, \xi, \varepsilon) = E$:

$$I = \int_\gamma y\,dx = I(E, \xi, \varepsilon). \tag{21}$$

If E exceeds all critical values of W in x, then γ is a simple closed curve. Since $\frac{\partial I}{\partial E} > 0$ for E large enough and $\varepsilon > 0$ small enough, we can define the inverse function of $I(E)$ so that E is a well-defined function of (I, ξ, ε), which we denote by K^0. The

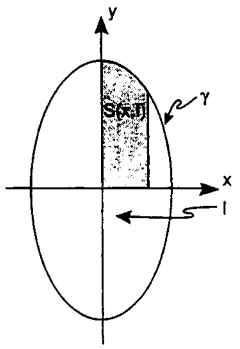

FIG. 3.1.

generating function S is then defined as the shaded area in Fig. 3.1 or, equivalently, as the solution of

$$K^0(I,\xi,\varepsilon) = H(x, S_x(x,I,\xi,\varepsilon),\xi,\varepsilon); \tag{22}$$

it is independent of the angle variable φ. As in [6] one verifies that

$$K^0(I,\xi,\varepsilon) = a(\xi)I^\beta + O(\varepsilon^\alpha) \tag{23}$$

with $\beta = \frac{2n+2}{n+2}$ and a positive function $a \in C^k(T^N)$. Now setting $\xi = \varepsilon\omega t$, we define the *time-dependent* symplectic transformation (20) by means of the generating function $S = S(x,I,\varepsilon\omega t,\varepsilon)$. It transforms the Hamiltonian system (18) into the system

$$K(\varphi,I,\varepsilon\omega t,\varepsilon) = K^0(I,\varepsilon\omega t,\varepsilon) + \frac{\partial}{\partial t}S(x,I,\varepsilon\omega t,\varepsilon) \tag{24}$$

on the phase space $(\varphi,I,t) \in S^1 \times \mathbf{R} \times \mathbf{R}$. Here $I \in \mathbf{R}$ varies in a bounded interval which is independent of ε; ε is small and $x = x(\varphi,I,\varepsilon\omega t,\varepsilon)$ in view of (20).

We denote the second term on the right-hand side by $K^1(\varphi,I,\varepsilon\omega t,\varepsilon)$, so that (24) becomes

$$K(\varphi,I,\varepsilon\omega t,\varepsilon) = K^0(I,\varepsilon\omega t,\varepsilon) + K^1(\varphi,I,\varepsilon\omega t,\varepsilon).$$

In what follows it will be crucial that

$$K^1(\varphi,I,\varepsilon\omega t,\varepsilon) = \varepsilon\omega\cdot\frac{\partial}{\partial\xi}S(x,I,\varepsilon\omega t,\varepsilon) = O(\varepsilon) \tag{25}$$

with all its (finitely many) derivatives in (φ,I,t). Here $\frac{\partial}{\partial\xi}S$ is the gradient of S with respect to ξ.

3.3. Choosing the symplectic angle as time. The Hamiltonian equations associated with the function K in (24) are, on the extended phase space $(\varphi,I,t) \in S^1 \times \mathbb{R} \times \mathbb{R}$, given by

$$\begin{aligned} \frac{d\varphi}{dt} &= K_I(\varphi,I,\varepsilon\omega t,\varepsilon),\\ \frac{dI}{dt} &= -K_\varphi(\varphi,I,\varepsilon\omega t,\varepsilon),\\ \frac{dt}{dt} &= 1. \end{aligned} \tag{26}$$

It is well known that if the time t and the "energy" K are chosen as the new conjugate variables and the angle φ is chosen as the new time variable, then (26) is transformed into an equation which is again Hamiltonian and belongs to a Hamiltonian function Q, which is the inverse of the function $I \mapsto K(\varphi, I, \varepsilon\omega t, \varepsilon)$. Indeed, from (23) we conclude that the partial derivative $K_I > 0$ if ε is small, and we can therefore define the transformation

$$\psi: \quad \begin{cases} q = t, \\ p = K(\varphi, I, \varepsilon\omega t, \varepsilon), \\ s = \varphi \end{cases} \tag{27}$$

from $(\varphi, I, t) \in S^1 \times \mathbb{R} \times \mathbb{R}$ into $(q, p, s) \in \mathbb{R} \times \mathbb{R} \times S^1$. Denote by $Q(\xi, p, \varphi, \varepsilon)$ the inverse function of $I \mapsto K(\varphi, I, \xi, \varepsilon)$, so that

$$p = K(\varphi, Q(\varepsilon\omega t, p, \varphi, \varepsilon), \varepsilon\omega t, \varepsilon). \tag{28}$$

Then the flow induced in the (q, p, s)-space by (26) is Hamiltonian with Q as the Hamiltonian function. This follows from $Id\varphi - Kdt = -(pdq - Qds)$ (see [23], [24]) or by a direct calculation which we now carry out. Abbreviating the vector field on the right-hand side of (26) by X, one readily verifies, using (27) and (28), that the transformed vector field is given by

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ s \end{pmatrix} = (d\psi)^{-1} X \circ \psi = \begin{pmatrix} 1 \\ -Q_p^{-1} Q_q \\ Q_p^{-1} \end{pmatrix},$$

where $Q = Q(\varepsilon\omega q, p, s, \varepsilon)$. Multiplying this vector field by the positive function Q_p we find as claimed

$$\begin{aligned} \frac{dq}{ds} &= Q_p(\varepsilon\omega q, p, s, \varepsilon), \\ \frac{dp}{ds} &= -Q_q(\varepsilon\omega q, p, s, \varepsilon), \end{aligned} \tag{29}$$

where the new time variable $s = \varphi$ is the old angle. One verifies, moreover, that

$$Q(\varepsilon\omega q, p, s, \varepsilon) = Q^0(\varepsilon\omega q, p, \varepsilon) + \varepsilon Q^1(\varepsilon\omega q, p, s, \varepsilon), \tag{30}$$

where

$$Q^0(\varepsilon\omega q, p, \varepsilon) = b(\varepsilon\omega q)p^\gamma + O(\varepsilon^\alpha) \tag{31}$$

with $\gamma = \frac{n+2}{2n+2}$ and a positive periodic function $b \in C^k(T^N)$.

3.4. Removing the time dependence in the dominant term. In order to relate the notation of the variables q, p, s of the Hamiltonian function (30) to their original meaning, we return to the old notation and replace the variables (q, p, s) and the Hamiltonian function Q by (t, K, ε) and I, so that the Hamiltonian (30) in this notation is

$$I(\varepsilon\omega t, K, \varphi, \varepsilon) = I^0(\varepsilon\omega t, K, \varepsilon) + \varepsilon I^1(\varepsilon\omega t, K, \varphi, \varepsilon), \tag{32}$$

where $I^0 = Q^0$ and $I^1 = Q^1$ are given by (30) and (31). The Hamiltonian equations now look as follows:

(33)
$$\frac{dt}{d\varphi} = I_K \ (\varepsilon\omega t, K, \varphi, \varepsilon),$$

$$\frac{dK}{d\varphi} = -I_t \ (\varepsilon\omega t, K, \varphi, \varepsilon).$$

Introducing $\varepsilon t = T$, these equations become

(34)
$$\frac{dT}{d\varphi} = \varepsilon\, I_K \ (\omega T, K, \varphi, \varepsilon),$$

$$\frac{dK}{d\varphi} = -\varepsilon I_T \ (\omega T, K, \varphi, \varepsilon)$$

and belong to the Hamiltonian function

$$\varepsilon\, I\ (\omega T, K, \varphi, \varepsilon).$$

We look for a symplectic transformation ψ of the form

(35)
$$\psi: \quad \begin{array}{rcl} \tau &=& T + u\,(T, K), \\ h &=& K + v\,(T, K), \end{array}$$

which is, in particular, independent of the (time) variable φ and transforms the Hamiltonian function εI into the following form:

(36)
$$\varepsilon\, \{I \circ \psi^{-1}\} = \varepsilon\, \{\Phi^0(h, \varepsilon) + \varepsilon\, \Phi^1(\omega\tau, h, \varphi, \varepsilon)\},$$

where the dominant term Φ^0 is independent of τ. To define this transformation we first define the leading term Φ^0 by taking the inverse function of $K \to I^0(\omega T, K)$, then averaging it over the torus T^N and taking the inverse again. Observe that $I^0_K > 0$ in the view of (31) provided that ε is sufficiently small. Therefore we can solve $I^0(\xi, K) = h$ for K and find a function $K^0(\xi, h)$ satisfying

(37)
$$I^0(\xi, K^0(\xi, h)) = h,$$

where K^0 is periodic in $\xi \in T^N$. Next, define the mean value over the torus by

(38)
$$[K^0](h) = \int_{T^N} K^0(\xi, h) d\xi.$$

Since $K^0_h = (I^0_K)^{-1} > 0$, the function $[K^0]$ has an inverse Φ^0 which thus satisfies

(39)
$$[K^0] \circ \Phi^0(h) = h.$$

In our notation we have neglected the dependence on ε. Clearly, $\Phi^0_h > 0$. This finishes the definition of Φ^0. Using this Φ^0, we shall next define the required symplectic transformation ψ in (35) implicitly by means of a generating function $\Sigma(\omega T, h)$, which is quasiperiodic in T,

(40)
$$\psi: \quad \begin{array}{rcl} \tau &=& T + \Sigma_h(\omega T, h), \\ K &=& h + \Sigma_T(\omega T, h). \end{array}$$

In order to achieve our aim (36) we have to solve the following equation for Σ:

(41) $$I^0(\omega T, h + \Sigma_T(\omega T, h)) \; = \; \Phi^0(h),$$

where the dependence on the parameter ε is again neglected. In view of (37) equation (41) is equivalent to

$$h + \Sigma_T(\omega T, h) \; = \; K^0(\omega T, \Phi^0(h)).$$

Therefore, the function $\Sigma(\xi, h)$ solves the following partial differential equation on T^N having the constant coefficients $(\omega_1, \dots, \omega_N) = \omega$:

(42) $$\sum_{j=1}^{N} \omega_j \, \frac{\partial}{\partial \xi_j} \Sigma\,(\xi, h) \; = \; K^0(\xi, \Phi^0(h)) - h.$$

Since, by our assumption, the frequencies ω satisfy the Diophantine conditions (5) and, by construction, the mean value over the torus of the right-hand side of (42) vanishes in view of (39), there is a unique solution $\Sigma(\xi, h)$ periodic in ξ and having vanishing mean value $[\Sigma](h) = 0$. Because of the well-known small divisor phenomenon, however, this solution loses derivatives, so that

(43) $$\Sigma \; \in \; C^{k-\tau}\,(T^N \times D)$$

if the right-hand side is in C^k, where the parameter τ in (44) is the same as in the Diophantine condition (5). This is well known and we refer to [10] and [16] for a proof. In view of

$$1 + \Sigma_{Th}\,(\omega T, h) = K^0_h(\omega T, \Phi^0(h)) \;\; \Phi^0_h(h)$$

$$= \frac{\Phi^0_h}{I^0_K} \; > \; 0\,,$$

the relation (40) indeed defines a symplectic transformation ψ of the form (35) . It is of class $C^{k-\tau-1}$; the extra loss of smoothness is a result of the differentiation in (40). Moreover, by the well-known properties of quasiperiodic functions proved in the book by Siegel and Moser on celestial mechanics [17, §36], one readily verifies that the functions u and v in the transformation ψ are quasiperiodic in T still with the same frequencies ω. The same conclusion follows for the functions representing the inverse transformation of ψ.

Recalling that $T = \varepsilon t$, we now replace τ by $\varepsilon\tau$ and arrive, in view of the Hamiltonian equations corresponding to the function (36), at the equations

(44) $$\frac{d\tau}{d\varphi} = \Phi_h \;\; (\omega\varepsilon\tau, h, \varphi, \varepsilon),$$

$$\frac{dh}{d\varphi} = -\Phi_\tau \;\; (\omega\varepsilon\tau, h, \varphi, \varepsilon).$$

The Hamiltonian function

(45) $$\Phi\;(\varepsilon\omega\tau, h, \varphi, \varepsilon) \; = \; \Phi^0(h, \varepsilon) + \varepsilon\Phi^1(\varepsilon\omega\tau, h, \varphi, \varepsilon)$$

is quasiperiodic in τ with frequencies $\varepsilon\omega$. Moreover, $\Phi(\xi, h, \varphi, \varepsilon)$ belongs to $C^{k-1-\tau}(T^N \times D \times S^1 \times \mathbf{R})$ and, by construction,

(46) $$\Phi^0\,(h, \varepsilon) \; = \; c\,h^\gamma + O(\varepsilon)$$

for a constant $c > 0$.

3.5. Back to the angle and action variables. Proceeding as in step (3.3), we next choose the variables $\varphi, I,$ and τ as the new position, momentum, and time variables. These variables then satisfy the Hamiltonian equations whose Hamiltonian function is the inverse function of $h \to \Phi(\varepsilon\omega\tau, h, \varphi, \varepsilon)$ denoted by $h(\varphi, I, \varepsilon\omega\tau, \varepsilon)$ and thus satisfying

$$\Phi\left(\varepsilon\omega\tau, h(\varphi, I, \varepsilon\omega\tau, \varepsilon)\right) = I.$$

Therefore, the Hamiltonian equations become, on the extended phase space $(\varphi, I, \tau) \in S^1 \times \mathbf{R} \times \mathbf{R}$,

$$\frac{d\varphi}{d\tau} = h_I\,(\varphi, I,\, \varepsilon\omega\tau, \varepsilon),$$

$$\frac{dI}{d\tau} = -h_\tau\,(\varphi, I,\, \varepsilon\omega\tau, \varepsilon).$$

The Hamiltonian function h is of class $C^{k-\tau-1}$, depends quasiperiodically on the time τ, and is of the form

(47) $$h\,(\varphi, I, \varepsilon\omega\tau, \varepsilon) = h^0\,(I, \varepsilon) + \varepsilon h^1\,(\varphi, I, \varepsilon\omega\tau, \varepsilon),$$

where h^0 is the inverse function of Φ^0, which thus satisfies

$$h^0\,(I, \varepsilon) = cI^\beta + O(\varepsilon^\alpha)$$

with constants $\beta = \frac{2n+2}{n+2}$, $\alpha > 0$, and $c > 0$.

3.6. Transformation into a system near an integrable one. In order to remove the dependence on φ in the $0(\varepsilon)$-terms of the Hamiltonian (46) we seek a time-dependent symplectic transformation $\psi : (\varphi, I) \to (x, y)$ between two annuli given by means of a generating function $S = S\,(\varphi, y, \varepsilon\omega\tau, \varepsilon)$, implicitly, via

(48) $$\psi : \quad \begin{array}{rcl} I &=& y + \varepsilon S_\varphi\,(\varphi, y), \\ x &=& \varphi + \varepsilon S_y\,(\varphi, y). \end{array}$$

Inserting (48) into (47) and expanding in ε leads to

$$\begin{aligned} &h^0\,(y + \varepsilon S_\varphi) \;+\; \varepsilon h^1\,(x + \varepsilon S_y\,, y + \varepsilon S_\varphi) \\ &\quad = h^0\,(y) \;+\; \varepsilon h^0_I\,(y) S_\varphi \;+\; \varepsilon h^1\,(x, y) \;+\; O\,(\varepsilon^2). \end{aligned}$$

To kill the angle dependence in the $O\,(\varepsilon)$ term above we require

$$h^0_I\,(y)\, S_\varphi \;+\; h^1\,(x, y, \varepsilon\omega\tau) = [h^1]\,(y, \varepsilon\omega\tau)$$

with the mean value over S^1 defined by

(49) $$[h^1]\,(y, \varepsilon\omega\tau) = \int_0^1 h^1\,(x, y, \varepsilon\omega\tau) dx,$$

and we find

$$S\,(\varphi, y, \varepsilon\omega\tau) = \frac{1}{h^0_I\,(y)} \int_0^\varphi \{[h^1] - h^1\} dx,$$

which is periodic in φ and quasiperiodic in τ. For the transformed Hamiltonian function $H\ (x,y,\varepsilon\omega\tau,\varepsilon) = h\circ\psi + \varepsilon S_\tau$ we therefore conclude

$$H\ (x,y,\varepsilon\omega\tau,\varepsilon) = h^0\ (y,\varepsilon) + \varepsilon[h^1]\ (y,\varepsilon\omega\tau,\varepsilon) + \varepsilon^2 h^2\ (x,y,\varepsilon\omega\tau,\varepsilon). \tag{50}$$

By repeating the same procedure but replacing εS by $\varepsilon^2 S$, of course with a different function S, the Hamiltonian (50) is transformed into a new Hamiltonian of the form

$$H\ (x,y,\varepsilon\omega\tau,\varepsilon) = h^0\ (y,\varepsilon) + \varepsilon[h^1]\ (y,\varepsilon\omega\tau,\varepsilon) + \varepsilon^2[h^2]\ (y,\varepsilon\omega\tau,\varepsilon) + \varepsilon^3 h^3\ (x,y,\varepsilon\omega\tau,\varepsilon). \tag{51}$$

Now, in order to remove the time dependence from the dominant part in (51) consisting of the first three terms, one again carries out step (3.3), then step (3.4), and then step (3.5) and finally arrives at the following time-dependent Hamiltonian function $H\ (x,y,\varepsilon\omega t,\varepsilon)$, which in action and angle variables $(x,y)\in S^1\times D$ for some open and bounded interval $D\subset\mathbf{R}^+$, is given by

$$H\ (x,y,\varepsilon\omega t,\varepsilon) = H_0\ (y,\varepsilon) + \varepsilon^3 H_1\ (x,y,\varepsilon\omega t,\varepsilon), \tag{52}$$

with

$$H_0\ (y,\varepsilon) = cy^\beta + O\ (\varepsilon^\alpha)$$

for positive constants c, α, and $\beta = \frac{2n+2}{n+2}$. The function H belongs to $C^{k-2\tau-4}$ and, moreover, is quasiperiodic in time t with the frequencies $\varepsilon\omega$. On the domain $(x,y,t)\in S^1\times D\times\mathbf{R}$ the system described by (52) turns out to be sufficiently near the integrable system, which is described by the Hamiltonian $H_0\ (y,\varepsilon)$ provided ε is small. This is the content of the next and last step in the proof of Theorem 1.

3.7. Existence of an invariant cylinder, proof of Theorems 1 and 2. We consider the Hamiltonian system (52) in $S^1\times D$, where D is a bounded interval of the positive real axis, H is periodic in x, and $\xi = (\xi_1,\dots,\xi_N)$:

$$H\ (x,y,\xi,\varepsilon) = H_0\ (y,\varepsilon) + \varepsilon^3 H_1\ (x,y,\xi,\varepsilon). \tag{53}$$

We are looking for quasiperiodic solutions having the frequencies $(\alpha,\varepsilon\omega)\in\mathbf{R}^{1+N}$, where ω are the prescribed frequencies of Theorem 1. In more geometric terms we look for a differentiable mapping

$$w : T^{1+N}\to S^1\times D, \tag{54}$$

$w\ (\theta,\xi) = (u\ (\theta,\xi), v\ (\theta,\xi))$, where $u\ (\theta,\xi)-\theta$ and $v\ (\theta,\xi)$ are periodic functions in θ and ξ, which maps the constant vector field V on T^{1+N}, given by

$$V:\ \begin{aligned}\dot\theta &= \alpha,\\ \dot\xi &= \varepsilon\omega\end{aligned} \tag{55}$$

into the given Hamiltonian vector field belonging to (53), thus satisfying

$$D_V\ w = dw\begin{pmatrix}\alpha\\ \varepsilon\omega\end{pmatrix} = J\nabla H\ (w\ (\theta,\xi),\xi) \tag{56}$$

for all $(\theta, \xi) \in T^{1+N}$. Here ∇ stands for the gradient with respect to the variables (x, y) and

$$D_V = \alpha \frac{\partial}{\partial \theta} + \varepsilon \sum_{j=1}^{N} \omega_j \frac{\partial}{\partial \xi_j},$$

(57)
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From (56) it then follows that a solution $(\theta(t), \xi(t)) = (\alpha t, \varepsilon\omega t)$ of V in (55) is mapped into the quasiperiodic solution $z(t) = w(\alpha t, \varepsilon\omega t)$ of the Hamiltonian system

(58)
$$\dot{z}(t) = J\nabla H(z(t), \varepsilon\omega t, \varepsilon).$$

One concludes, in particular, that the cylinder

(59)
$$\hat{w} : S^1 \times \mathbf{R} \to S^1 \times D \times \mathbf{R},$$

defined by $\hat{w}(\theta, t) = (w(\theta, \varepsilon\omega t), t)$, is tangential to $(J\nabla H(x, y, \varepsilon\omega t), 1)$, the Hamiltonian vector field in the phase space $S^1 \times D \times \mathbf{R}$. The solutions on this cylinder are, moreover, quasiperiodic. The required map w is, in view of (56), a solution of the nonlinear partial differential equation

(60)
$$D_V w = J\nabla H(w, \xi, \varepsilon).$$

In the special case of the integrable system defined by $H_0(y, \varepsilon)$, which does not depend on the torus variables $(\theta, \xi) \in T^{1+N}$, the solutions w of (60) are simply the injection mappings

(61)
$$w : T^{1+N} \to S^1 \times D, \quad w(\theta, \xi) = (\theta, y),$$

where y is determined by the vector field $V = (\alpha, \varepsilon\omega)$ via

(62)
$$\alpha = \frac{\partial H_0}{\partial y}(y, \varepsilon).$$

If $\varepsilon > 0$ and small, then the system H is a perturbation of this integrable system and we shall apply a well-known existence statement of Moser [18] in order to guarantee solutions w of (60) nearby.

First we observe that, by construction, the function H in (52) satisfies

(63)
$$C < \frac{\partial^2 H_0}{\partial y^2}(y, \varepsilon) < C^{-1}, \quad y \in D$$

for a positive constant $C > 0$, which is independent of ε for small ε. This is the twist condition. Consequently, if the prescribed frequencies $\omega \in \mathbf{R}^N$ satisfy the Diophantine conditions (5) of the theorem, then we find, for every given $\varepsilon > 0$, a point $y \in D$ such that $\alpha = (\partial H_0/\partial y)(y, \varepsilon)$ satisfies

(64)
$$|\alpha \cdot k + \langle \varepsilon\omega, j \rangle| \geq \varepsilon\gamma (|k| + |j|)^{-\tau}$$

for all $(k,j) \in \mathbf{Z}^{1+N} \backslash \{0\}$ with the constants $\gamma > 0$ and $\tau > N$ as in (5). Indeed, one readily verifies, using $\tau > N$, that in every finite interval I the complement of those real numbers α in I which fail the estimates (64) are a set of Lebesgue measure $O(\varepsilon)$. Secondly, the Hamiltonian function H is sufficiently smooth: $H \in C^l$ for $l > 2\tau + 2$. Indeed, from step (3.6) we have, by construction, $H \in C^{k-2\tau-4}$ and, by assumption, $k > 4\tau + 6$. Thirdly, the perturbation is sufficiently small in the sense that

$$\left(\frac{1}{\varepsilon\gamma}\right)^2 \|H - H_0\|_{C^l} = \frac{\varepsilon}{\gamma^2}\|H_1\|_{C^l} = O(\varepsilon). \tag{65}$$

In view of (63), (64), and (65) we can apply Moser's theorem in [18] together with its improvements from Salamon in [20] and Salamon and Zehnder in [16], which remove the analycity requirement for the unperturbed system. We conclude that, for $0 < \varepsilon \leq \varepsilon^*$ small and $\alpha = (\partial H_0/\partial y)(y,\varepsilon)$ satisfying (64), there exists a solution $w = w_\varepsilon$ of (60). Moreover, this solution $w_\varepsilon = (u_\varepsilon, v_\varepsilon)$ satisfies

$$\begin{aligned} \|u_\varepsilon(\theta,\xi) - \theta\|_{C^1} &= O(\varepsilon), \\ \|v_\varepsilon(\theta,\xi) - y\|_{C^1} &= O(\varepsilon), \end{aligned} \tag{66}$$

so that w_ε is indeed close to the map $w = w_0$ in (61) for the integrable system. It belongs to the same α.

Summarizing, for every $\varepsilon > 0$ sufficiently small we have constructed an invariant cylinder (59). Going back to the original coordinates we conclude, in view of the scaling in step 1, that to every initial condition $(x(0), \dot{x}(0)) \in \mathbf{R}^2$ there is an invariant cylinder as described in the introduction containing the corresponding solution $(x(t), \dot{x}(t))$ of equation (1) in its interior, so that $\sup\{x(t)^2 + \dot{x}(t)^2, t \in \mathbf{R}\} < \infty$. This finishes the proof of Theorem 1.

We remark that, in order to apply the above small denominator techniques to our problem at hand, one simply extends the Hamiltonian system by considering the function

$$\hat{H}(x,\xi,y,\eta) = \varepsilon\langle\omega,\eta\rangle + H(x,y,\xi,\varepsilon)$$

on the extended phase space $T^{1+N} \times \mathbf{R}^{1+N}$ with its standard symplectic structure. The integrable part of $\hat{H}$ is then given by

$$\hat{H}_0(y,\eta,\varepsilon) = \varepsilon\langle\omega,\eta\rangle + H_0(y,\varepsilon).$$

The distinguished invariant torus of this integrable system, defined by $T^{1+N} \times \{y,\eta\}$ for $\alpha = (\partial H_0/\partial y)(y,\varepsilon)$ and $\eta = 0$, which has the frequencies $(\alpha, \varepsilon\omega)$, is then continued under the perturbation. For the existence proof of this continuation one simply applies the standard transformation technique, restricting, however, the symplectic transformations used to the subgroup of those transformations leaving the ξ variables fixed.

4. Proof of Theorem 3 (on general potentials). In this section we shall sketch the proof of Theorem 3. The many tedious technical details are the same as in [15] and the proof of Theorem 1 and will be omitted. We first carry out the formal steps which put $\ddot{x} + V_x(x,\omega t) = 0$ into a suitable normal form in the region $x^2 + \dot{x}^2 \geq C$ in $\mathbf{R}^2$ for $C > 0$ large. Afterwards we follow up with the estimates.

4.1. The formal normal form. In contrast to the polynomial case of Theorem 1 we do not rescale until later.

Step 1. We first introduce the action and angle variables $(x, \dot{x}, t) \to (\theta, I, t)$ by freezing a value t and assigning to (x, y, t) the triple (θ, I, t) as follows:

$$\begin{aligned} \theta &= S_I\,(x, I, \omega t), \\ y &= S_x\,(x, I, \omega t), \end{aligned} \tag{67}$$

where

$$S\,(x, I, \omega t) \;=\; \int_0^x y dx$$

is the integral taken along the level curve $\frac{1}{2}\,y^2 \;+\; V\,(x, \omega t) = \text{const}$, which encloses the area I in the (x, y)-plane. The resulting Hamiltonian in the (θ, I, t) variables is then

$$H\,(\theta, I, \omega t) \;=\; H_0\,(I, \omega t) \;+\; H_1\,(\theta, I, \omega t) \tag{68}$$

with

$$H_1 \;=\; S_t \;\dot{=}\; \omega \cdot \frac{\partial}{\partial \xi} S\,(\theta, I, \omega t).$$

Step 2. Now, proceeding as in (3.3) we choose t, H, θ and $I \;=\; I\;(\omega t, H, \theta)$ as the new position, momentum, time, and Hamiltonian function, respectively. Here $I\;(\omega t, H, \theta)$ is the inverse function of $I \;\mapsto\; H\;(\theta, I, \omega t)$. Defining $I_0\;(\omega t, H)$ as the inverse of $I \;\mapsto\; H_0\;(I, \omega t)$ we rewrite the Hamiltonian describing the system in the form

$$I\,(\omega t, H, \theta) \;=\; I_0\,(\omega t, H) \;+\; I_1\,(\omega t, H, \theta), \tag{69}$$

thus defining I_1 with θ playing the role of the time.

Step 3. Removing the t-dependence in the "leading" term H_0, following §3.4 we arrive at the equivalent Hamiltonian system defined by

$$J\,(\omega\tau, h, \theta) \;=\; J_0\,(h) \;+\; J_1\,(\omega\tau, h, \theta). \tag{70}$$

Step 4. Proceeding as in §3.5, we go back to the variables (θ, J, τ) as the new position, momentum, and time and arrive at the new system with the Hamiltonian

$$h\,(\theta, J, \omega\tau) \;=\; h_0\,(J) \;+\; h_1\,(\theta, J, \omega\tau), \tag{71}$$

where h is the inverse function of $h \mapsto J\;(\omega\tau, h, \theta)$ and h_0 is the inverse of $h \mapsto J_0\;(h)$.

Step 5. In order to work on a bounded interval for the action variable we rescale (θ, J) into (x, y) by setting

$$x \;=\; \theta, \qquad y \;=\; \varepsilon J, \tag{72}$$

where $\varepsilon > 0$ is small and y varies in a bounded interval $D \;\subset \mathbf{R}^+$. The new variables satisfy the Hamiltonian system corresponding to the Hamiltonian function

$$\varepsilon\, h\,(x, y, \omega\tau) \;=\; \varepsilon\, h_0\,\left(\frac{y}{\varepsilon}\right) \;+\; \varepsilon\, h_1\,\left(x, \frac{y}{\varepsilon}, \omega\tau\right). \tag{73}$$

Rescaling the time by

$$\tau = \frac{t}{\varepsilon h_0(\frac{1}{\varepsilon})} \tag{74}$$

and introducing the abbreviation $T(\varepsilon) = \varepsilon h_0(\frac{1}{\varepsilon})$, we set

$$\Omega = \Omega(\varepsilon) = \frac{\omega}{T(\varepsilon)} \tag{75}$$

and arrive at the new Hamiltonian system given by

$$\hat{h}(x, y, \Omega t, \varepsilon) = \hat{h_0}(y, \varepsilon) + \varepsilon^\beta \hat{h_1}(x, y, \Omega t, \varepsilon) \tag{76}$$

with the functions $\hat{h_0}$ and $\hat{h_1}$ defined by

$$\hat{h_0}(y, \varepsilon) = \frac{1}{h_0(\frac{1}{\varepsilon})} h_0\left(\frac{y}{\varepsilon}\right),$$

$$\hat{h_1}(x, y, \Omega t, \varepsilon) = \frac{1}{\varepsilon^\beta h_0(\frac{1}{\varepsilon})} h_1\left(x, \frac{y}{\varepsilon}, \Omega t\right).$$

The constant $\beta > 0$ will be chosen later such that $\hat{h_1}$, together with all its finitely many derivatives, is bounded independently of $\varepsilon > 0$.

Step 6. After K further symplectic transformations as in §3.6, we find the following normal form for the Hamiltonian function on an annulus A with coordinates x and y:

$$H(x, y, \Omega t, \varepsilon) = H_0(y, \varepsilon) + \varepsilon^{K\beta} H_K(x, y, \Omega t, \varepsilon), \tag{77}$$

where H is periodic in x and quasiperiodic in t with the frequencies $\Omega = \Omega(\varepsilon)$ as defined in (75). All the transformations result in the loss of $2\tau + 3 + K$ derivatives.

4.2. Estimates. In order to apply the theorems of existence of quasiperiodic solutions as in §3.7 for the flow of the Hamiltonian system given by H in (77), we need the following conditions (i)–(iii) to be satisfied:

(i) There exist constants $0 < C_1 < C_2$ independent of $\varepsilon > 0$ such that

$$C_1 \leq \left(\frac{\partial}{\partial y}\right)^2 H_0(y, \varepsilon) \leq C_2 \tag{78}$$

for all $y \in D$, where D is a bounded interval in $\mathbf{R}^+$.

(ii) For every $\varepsilon > 0$ there is a $y \in D$ such that $\alpha = \frac{\partial}{\partial y} H_0(y, \varepsilon)$ satisfies the Diophantine conditions

$$|\alpha k + \langle \Omega, j \rangle| \geq \gamma |\Omega| (|k| + |j|)^{-\tau} \tag{79}$$

for all $(k, j) \in \mathbf{Z} \times \mathbf{Z}^N \setminus \{0\}$, where $\Omega = \Omega(\varepsilon)$ is given by (75).

(iii) The perturbation satisfies

$$\left(\frac{1}{\gamma|\Omega|}\right)^2 \|\varepsilon^{K\beta} H_K\|_{C^l} = O(\varepsilon^\sigma) \tag{80}$$

for some $l > 2\tau + 2$ and all $\varepsilon > 0$ with a constant $\sigma > 0$.

The following two propositions will allow the verification of (78)–(80). The first proposition is a restatement of Theorems 3 and 4 in [15, p. 47].

PROPOSITION 1. *If the potential $V(x,\xi)$ satisfies the assumptions of Theorem 3 with $|\alpha|+|k| \le d$ then the Hamiltonian function (68) satisfies for I large the following estimates: There are positive constants $C_{k,\alpha}$ and $\delta = \delta(\mu,b,d)$ such that*

$$|\partial_\xi^\alpha \partial_I^k H_1(\theta,I,\xi)| \le C_{k,\alpha}\, I^{-k-\delta}\, H_0(I,\xi), \quad |\alpha|+|k| \le d-1. \tag{81}$$

Moreover, the function $I_0(\xi,H)$ in (69) satisfies for large H,

$$|\partial_\xi^\alpha \partial_H^k I_0(\xi,H)| \le C_{k,\alpha}\, H^{-k}\, I_0(\xi,H), \quad |\alpha|+|k| \le d-1 \tag{82}$$

and for $0 \le k \le 2$,

$$H^{-k}\, I_0(\xi,H) \le C\, |\partial_H^k I_0(\xi,H)| \tag{83}$$

for a constant $C > 0$.

The next proposition is a restatement of Theorem 5.1 in [15].

PROPOSITION 2. *If estimates (81)–(83) hold true then the Hamiltonian function (71) denoted by $J(\xi,h,\theta) = J_0(h) + J_1(\xi,h,\theta)$ satisfies the following estimates for h large: There are positive constants $C, C_k, C_{k,\alpha}, a, a_1$, and σ, such that*

$$\frac{1}{C}\, h^{-a} \le \left|\frac{\partial}{\partial h}\, J_0(h)\right| \quad \textit{and} \quad \frac{1}{C}\, h^{a_1} \le J_0(h), \tag{84}$$

$$\frac{1}{C}\, h^{-k}\, J_0(h) \le \left|\left(\frac{\partial}{\partial h}\right)^k J_0(h)\right| \quad \textit{for } 0 \le k \le 2, \tag{85}$$

$$\left|\left(\frac{\partial}{\partial h}\right)^k J_0(h)\right| \le C_k\, h^{-k}\, |J_0(h)|, \quad |k| \le d-2, \tag{86}$$

$$|\partial_\xi^\alpha \partial_h^k J_1(\xi,h,\theta)| \le C_{k,\alpha}\, h^{-k}\, J_0(h)^{1-\sigma}, \quad |\alpha|+|k| \le d-2. \tag{87}$$

By using the techniques developed in [15], one can follow these estimates through all our coordiante transformations in order to verify (78) and (80). In particular, estimates (84)–(87) imply the existence of constants $\beta = \beta(b,\mu,d)$ and $\mu_0 = \mu_0(b)$ such that

$$\|H_K\|_{C^{d-2\tau-3-K}} = O(1).$$

Furthermore, there exists $\alpha = \alpha(b) > 0$ such that

$$\varepsilon h_0\left(\frac{1}{\varepsilon}\right) = O(\varepsilon^{-\alpha}).$$

To verify (80) we combine the last two estimates

$$\left(\frac{1}{\gamma|\Omega|}\right)^2 \|\varepsilon^K H_K\|_{C^{d-2\tau-3-K}} = O\left(\varepsilon^{K\beta-2\alpha}\right),$$

and it remains to show that $K\beta(b,\mu,d)-2\alpha>0$ and $d-2\tau-3-K>2\tau+2$ hold for some $K>0$ or, equivalently,

$$2\frac{\alpha(b)}{\beta(b,\mu,d)}<K<d-4\tau-5.$$

For such K to exist it suffices to verify that

$$d-4\tau-5\frac{2\alpha(b)}{\beta(b,\mu,d)}\geq 1 \quad \text{and} \quad \beta(b,\mu,d)>0. \tag{88}$$

First, $\beta(b,0,d)=\beta(b)>0$ is d-independent, according to the arguments in [15], and $\beta(b,\mu,d)$ can be chosen as continuous in μ for $\mu\geq 0$. Thus, choosing

$$d \;=\; 4\tau+7+2\frac{\alpha(b)}{\beta(b)}$$

guarantees that (88) holds for all $\mu\in[0,\mu_0(b)]$, where $\mu_0(b)>0$ can be estimated explicitly. With this choice of d there exists K with which the smallness condition (80) is satisfied. This finishes the sketch of the proof of Theorem 3.

5. Proof of Theorem 4 (unbounded solutions). The idea is to create $p(t)$, giving a particular solution $y(t)$ of (11) a "helping kick" to the right each time the solution passes through the interval $-1\leq x\leq 1$ from -1 to $+1$, and make $p(t)=0$ at all other times. With such a $p(t)$ the energy along the solution will increase during each passage from -1 to $+1$ while remaining constant between consecutive passages. If the "kicks" do not weaken too much with the number of passages, the errors will grow without bound. The precise construction is as follows: We first consider an auxiliary nonconservative system

$$\ddot{y}+y^3=f(y,\dot{y}) \tag{89}$$

for a smooth function $f\in C^\infty(\mathbf{R}^2)$ defined by

$$f(y,\dot{y})=h(y)\cdot\frac{g(\dot{y})}{\dot{y}^r}, \tag{90}$$

where $r\in\mathbf{N}$ is as it was in the statement of the theorem and h and $g\in C^\infty(\mathbf{R})$ satisfy

$$h\,(y)\begin{cases}=0 & \text{if} \quad |y|\geq 1,\\ >0 & \text{if} \quad |y|<1,\end{cases}$$

$$g\,(\dot{y})\begin{cases}=0 & \text{if} \quad \dot{y}\leq 0,\\ >0 & \text{if} \quad 0<\dot{y}<1,\\ =1 & \text{if} \quad \dot{y}\geq 1.\end{cases} \tag{91}$$

Now, if $y(t)$ is a fixed unbounded solution of (89), we can define $p\,(t)=f\,(y\,(t),\dot{y}\,(t))$ and consider equation (11) with this forcing term p. The function $y\,(t)$ then solves both the conservative equation (11) and the nonconservative equation (89). It turns out that all nontrivial solutions of the latter are unbounded.

and it remains to show that $K\beta(b,\mu,d) - 2\alpha > 0$ and $d - 2\tau - 3 - K > 2\tau + 2$ hold for some $K > 0$ or, equivalently,

$$2\frac{\alpha(b)}{\beta(b,\mu,d)} < K < d - 4\tau - 5.$$

For such K to exist it suffices to verify that

$$d - 4\tau - 5\frac{2\alpha(b)}{\beta(b,\mu,d)} \geq 1 \quad \text{and} \quad \beta(b,\mu,d) > 0. \tag{88}$$

First, $\beta(b,0,d) = \beta(b) > 0$ is d-independent, according to the arguments in [15], and $\beta(b,\mu,d)$ can be chosen as continuous in μ for $\mu \geq 0$. Thus, choosing

$$d = 4\tau + 7 + 2\frac{\alpha(b)}{\beta(b)}$$

guarantees that (88) holds for all $\mu \in [0, \mu_0(b)]$, where $\mu_0(b) > 0$ can be estimated explicitly. With this choice of d there exists K with which the smallness condition (80) is satisfied. This finishes the sketch of the proof of Theorem 3.

5. Proof of Theorem 4 (unbounded solutions). The idea is to create $p(t)$, giving a particular solution $y(t)$ of (11) a "helping kick" to the right each time the solution passes through the interval $-1 \leq x \leq 1$ from -1 to $+1$, and make $p(t) = 0$ at all other times. With such a $p(t)$ the energy along the solution will increase during each passage from -1 to $+1$ while remaining constant between consecutive passages. If the "kicks" do not weaken too much with the number of passages, the errors will grow without bound. The precise construction is as follows: We first consider an auxiliary nonconservative system

$$\ddot{y} + y^3 = f(y, \dot{y}) \tag{89}$$

for a smooth function $f \in C^\infty(\mathbf{R}^2)$ defined by

$$f(y,\dot{y}) = h(y) \cdot \frac{g(\dot{y})}{\dot{y}^r}, \tag{90}$$

where $r \in \mathbf{N}$ is as it was in the statement of the theorem and h and $g \in C^\infty(\mathbf{R})$ satisfy

$$h(y) \begin{cases} = 0 & \text{if} \quad |y| \geq 1, \\ > 0 & \text{if} \quad |y| < 1, \end{cases}$$

$$g(\dot{y}) \begin{cases} = 0 & \text{if} \quad \dot{y} \leq 0, \\ > 0 & \text{if} \quad 0 < \dot{y} < 1, \\ = 1 & \text{if} \quad \dot{y} \geq 1. \end{cases} \tag{91}$$

Now, if $y(t)$ is a fixed unbounded solution of (89), we can define $p(t) = f(y(t), \dot{y}(t))$ and consider equation (11) with this forcing term p. The function $y(t)$ then solves both the conservative equation (11) and the nonconservative equation (89). It turns out that all nontrivial solutions of the latter are unbounded.

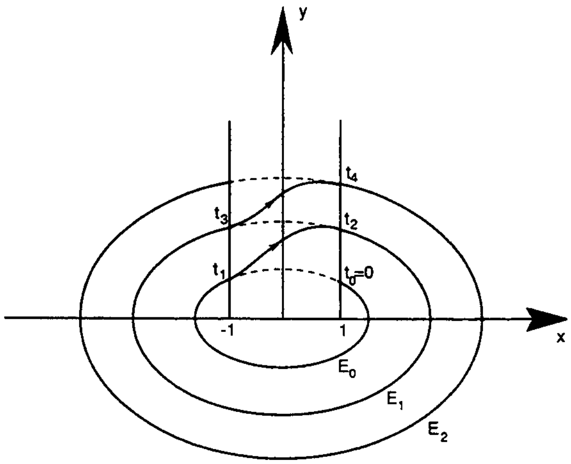

FIG. 5.1.

The energy

$$E\ (t) := \frac{1}{2}\dot{y}\ (t)^2 + \frac{1}{4}y\ (t)^4 \tag{92}$$

along a given solution $y\ (t)$ of (89) satisfies

$$\frac{d}{dt}\ E\ (t)\ =\ \dot{y}\ f\ (y, \dot{y}), \tag{93}$$

which is > 0 if $\dot{y} > 0$ and $|y| < 1$ and 0 otherwise.

To be specific, in the following we consider the solution $y(t)$ with initial values $y\ (0) = \dot{y}\ (0) = 1$. Referring to Fig. 5.1, the motion of $(y, \dot{y})$ in the $\mathbf{R}^2$-plane follows the energy curve in $\mathbf{R}^2$ given by $\frac{1}{2}\dot{y}^2 + \frac{1}{4}y^4 = \frac{3}{4}$ in the clockwise direction until the point $(-1, 1)$ is reached at some time $t_1 > t_0 = 0$. At some later moment $t_2 > t_1$ the point $(y, \dot{y})$ will cross the right boundary $\{y = 1, \dot{y} > 1\}$ of the strip. Indeed, this follows by comparison with the equation $\ddot{y} + y^3 = 0$ from the fact that $f(y, \dot{y}) > 0$ in the strip. Furthermore, $E\ (t_2) > E\ (t_1)$ by (93), so that the point will start moving along a larger energy curve after having crossed the strip.

Denoting the sequence of consecutive crossings of the vertical boundaries $\{y = \pm 1, \dot{y} > 0\}$ by $0 = t_0 < t_1 < t_2 < \cdots$, and the sequence of corresponding energy values between consecutive crossings by

$$E_n = E\ (t), \quad t_{2n} \le t \le t_{2n+1},$$

$n = 0, 1, 2, \ldots,$ we claim that

$$\lim_{n \to \infty} E_n = \lim_{t \to \infty} E\ (t) = \infty. \tag{94}$$

Indeed, integration of (93) gives

$$\begin{aligned} E_{n+1} - E_n &= \int_{t_{2n+1}}^{t_{2n+2}} \dot{y} f\ (y, \dot{y}) dt \\ &= \int_{t_{2n+1}}^{t_{2n+2}} h\ (y) \frac{1}{\dot{y}^{r-1}} dt = \int_{-1}^{1} h\ (y) \cdot \frac{1}{\dot{y}^r} dy. \end{aligned} \tag{95}$$

The inequalities

$$E_n < E\,(t) < E_{n+1} \text{ for } t_{2n+1} < t < t_{2n+2}$$

result, in view of $|y\,(t)| < 1$ and $E_n > \frac{1}{2}$, in

$$\sqrt{E_n} \leq \dot{y}\,(t) \leq \sqrt{2}\sqrt{E_{n+1}} \quad \text{for } t_{2n+1} \leq t \leq t_{2n+2}. \tag{96}$$

From (95) and (96) we conclude

$$\frac{a}{E_{n+1}^{r/2}} < E_{n+1} - E_n < \frac{b}{E_n^{r/2}}, \quad n = 0, 1, 2, \dots \tag{97}$$

for two constants $0 < a < b$. Consequently, E_n is a strictly monotone increasing sequence; moreover, $\lim E_n = \infty$. Indeed, if $\lim E_n = E_\infty < \infty$ then taking the limit in (97) leads to the contradiction $0 < a/E_\infty^{r/2} = 0$. We have proved claim (94) and conclude, in view of (92), that the solution $y\,(t)$ of (89) is unbounded.

In order to prove claim (10) in Theorem 4 for $p\,(t) = f\,(y\,(t), \dot{y}\,(t))$, we note first that, for $t_{2n+1} \leq t \leq t_{2n+2}$, we have $\dot{y}\,(t) > 0$ and thus

$$p\,(t) = h\,(y)\dot{y}^{-r},$$

while $p\,(t) = 0$ otherwise. Since $\lim E_n = \infty$, estimate (96) implies $\lim_{t\to\infty} p\,(t) = 0$.

Similarly, differentiating p and observing that $y\,(t)$ satisfies equation (89), one readily verifies the estimate

$$|D^j p\ \ (t)| \leq C(h)\frac{1}{\dot{y}\,(t)^{r-j}}, \quad \text{if } t_{2n+1} < t < t_{2n+2}, \tag{98}$$

while $D^j p\,(t) = 0$ otherwise, so that $\lim_{t\to\infty} D^j p\,(t) = 0$ if $1 \leq j \leq r-1$. Here, the constant $C\,(h)$ depends only on h and its derivatives and, moreover, $C\,(\varepsilon h) \leq \varepsilon C_1\,(h)$ for ε small. Replacing $h\,(y)$ by $\varepsilon h\,(y)$, we get the required estimate (10) for the forcing $p\,(t)$.

Finally, we prove estimates (12) and (13). Since, by (94) and (97), $c_1 < E_{n+1}/E_n < c_2$ for two positive constants $c_1 < c_2$, we have

$$\frac{a}{E_n^{r/2}} < E_{n+1} - E_n < \frac{b}{E_n^{r/2}} \tag{99}$$

for two constants $0 < a < b$, which are different from the previous constants. To estimate E_n we define the continuous function $\eta\,(t)$ by linearly interpolating E_n:

$$\eta\,(t) := (t-n)E_{n+1} + (1-t+n)E_n$$

if $n \leq t \leq n+1$, so that $E_n = \eta\,(n)$. Thus $\eta\,(t)$ is a strictly monotone increasing function whose right derivative D_r satisfies, in view of (99),

$$\frac{a}{\eta\,(t)^{r/2}} \leq D_r\eta\,(t) \leq \frac{b}{\eta\,(t)^{r/2}} \tag{100}$$

for all $t \geq 0$; this is the differential inequality which interpolates (99). Comparison with the solutions $\xi_a\,(t)$ and $\xi_b\,(t)$ of the two equations

$$\dot{\xi} = \frac{\alpha}{\xi^{r/2}} \quad \text{with } \alpha = a \text{ and } \alpha = b,$$

with initial conditions $\xi_a(0) = \eta(0) = \xi_b(0)$, leads to $\xi_a(t) \leq \eta(t) \leq \xi_b(t)$ for $t \geq 0$. Consequently, there are constants $0 < A < B$ such that

$$A\, n^{\frac{1}{1+r/2}} \leq E_n \leq B\, n^{\frac{1}{1+r/2}} \tag{101}$$

for $n = 0, 1, 2, \ldots$. In order to relate n with t_n we note that

$$T(E_{n+1}) < t_{2n+2} - t_{2n} < T(E_n), \tag{102}$$

where $T(E)$ denotes the period of the solutions of $\ddot{x} + x^3 = 0$ with energy $E = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$. A simple calculation gives $T(E) = \tau E^{-1/4}$ for some constant $\tau > 0$. Using this in (102), we conclude from (101) that

$$\frac{a}{n^{\frac{1}{2r+4}}} < t_{2n+2} - t_{2n} < \frac{b}{n^{\frac{1}{2r+4}}} \tag{103}$$

for two constants $0 < a < b$, which are different from the ones in previous formulas. Adding up inequalities (103), we obtain the estimate

$$c\, n^{\frac{2r+3}{2r+4}} < t_n < C\, n^{\frac{2r+3}{2r+4}} \tag{104}$$

for two constants $0 < c < C$. Recalling the definition of $E(t)$, we conclude from (101) and (104) that

$$a\, t^{\frac{4}{2r+3}} < E(t) < b\, t^{\frac{4}{2r+3}}, \quad t \geq 1 \tag{105}$$

for two constants $0 < a < b$. Finally, in view of (96) and (98) we now conclude

$$|D^j p(t)| \leq c_1 \frac{1}{\sqrt{E(t)}^{\,r-j}} \leq c_2 \frac{1}{t^{\frac{2(r-j)}{2r+3}}}$$

as claimed in the theorem. This finishes the proof of Theorem 4.

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