The Cauchy-Schwarz Inequality via Springs

MATHEMATICAL CURIOSITIES
By Mark Levi

Here I present a different physical implementation of the idea in [1][1] and [2]; the mathematical portion is exactly the same except for the notations, but I still present it for the sake of self-sufficiency. Let us connect n springs end-to-end, as shown in Figure 1 for $n=3$. Initially, we forcibly hold the connections at some arbitrary positions. Then we release them and let the system settle into the equilibrium configuration. In the process, potential energy decreases:

$$P_{\text{old}} \geq P_{\text{new}}. \qquad (1)$$

The equality holds if and only if the system was already in equilibrium at the outset. I claim that (1) is the Cauchy-Schwarz inequality (in disguise) if the springs are Hookean, i.e., if the tension of the ith spring is in direct proportion to its length: $F_i = k_i L_i$. Indeed, since a Hookean spring's potential energy is $\frac{1}{2}kL^2 = \frac{1}{2}\lambda F^2$—where $\lambda = k^{-1}$ is the spring's "laxness"—(1) amounts to

$$\sum \lambda_i F_i^2 \geq \bar{F}^2 \sum \lambda_i, \qquad (2)$$

where $\bar{F}$ is the common tension of all the springs when the system is in equilibrium. But

$$\bar{F} = \frac{\sum \lambda_i F_i}{\sum \lambda_i}$$

is the weighted average; I omit the verification of this fact, which involves showing that *"laxnesses" add for springs connected in series*.[2] Substituting this expression into (2) gives

$$\sum \lambda_i \sum \lambda_i F_i^2 \geq \left(\sum \lambda_i F_i\right)^2.$$

By setting $\lambda_i = x_i^2$ and $\lambda_i F_i^2 = y_i^2$, we get the familiar form of the Cauchy-Schwarz inequality. And if the springs are non-Hookean, with $F = kL^p$ and $p \neq 1$, then (1) amounts to a Holder inequality via an essentially verbatim repetition of the argument in [3].

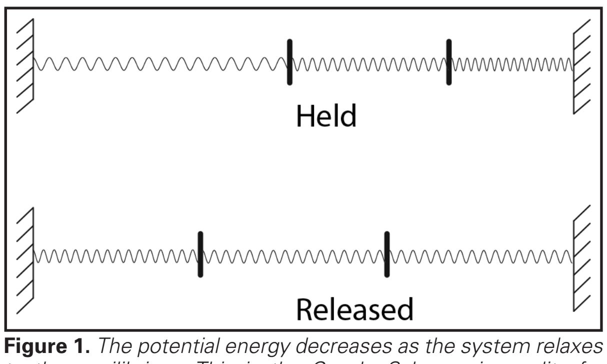

Figure 1. *The potential energy decreases as the system relaxes to the equilibrium. This is the Cauchy-Schwarz inequality for Hookean springs and the Holder's inequality for "polynomial" springs with $F = kL^p$. Figure courtesy of Mark Levi.*

References

[1] Levi, M. (2019). The Cauchy-Schwarz inequality and a paradox/puzzle. *SIAM News, 52*(9), p. 11.

[2] Levi, M. (2020). A water-based proof of the Cauchy-Schwartz inequality. *Am. Math. Month., 127*(6), p. 572.

[3] Levi, M., & Tokieda, T. (2020). A communicating-vessels proof of Holder's inequality. *Am. Math. Month*. In press.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.

[1] https://sinews.siam.org/Details-Page/the-cauchyschwarz-inequality-and-aparadoxpuzzle

[2] This is an analog of resistance additivity in electric circuits.