Foucault's Pendulum with a Twist

MATHEMATICAL CURIOSITIES

By Mark Levi

While teaching a mechanics course, I stumbled upon the following amusing observation. It is well known that small-amplitude trajectories of a pendulum are approximately ellipses (see Figure 1). Indeed, the linearized equations for the (x, y) coordinates of the bob are

$$\begin{aligned} \ddot{x} &= -\omega^2 x \\ \ddot{y} &= -\omega^2 y, \end{aligned} \qquad (1)$$

or

$$\ddot{\mathbf{r}} = -\omega^2 \mathbf{r}, \;\; \mathbf{r} = \langle x, y \rangle,$$

with $\omega^2 = g/L$; g is the gravitational acceleration and L is the length of the pendulum. The general solution is $x = a\cos\omega t + b\sin\omega t$, $y = c\cos\omega t + d\sin\omega t$. This is a parametric representation of an ellipse centered at the origin. Indeed, (x, y) is the image of the unit circle $(\cos\omega t, \sin\omega t)$ under the linear map whose matrix has elements a, b, c, d. Figure 2 illustrates the three types of motions.

Now let's put ourselves in a frame centered at the origin and rotating with angular velocity ω, where ω is the same as above: the frequency of the pendulum. *How will the elliptical motions of the pendulum look in this new frame?*

Perhaps surprisingly, the answer is *circular, and with a constant speed*. Moreover, the angular velocity of these circular motions is 2ω, twice the frequency of the pendulum (see Figure 3). The circle passes through the origin precisely if the angular momentum is zero. And the circle in the rotating frame is centered at the origin if the ellipse in the inertial frame is a circle.

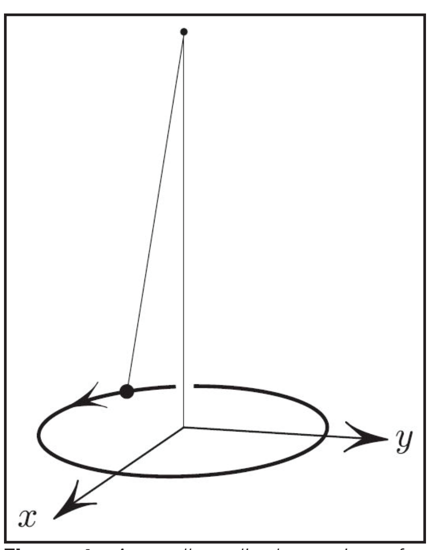

Figure 1. *A small-amplitude motion of a spherical pendulum.*

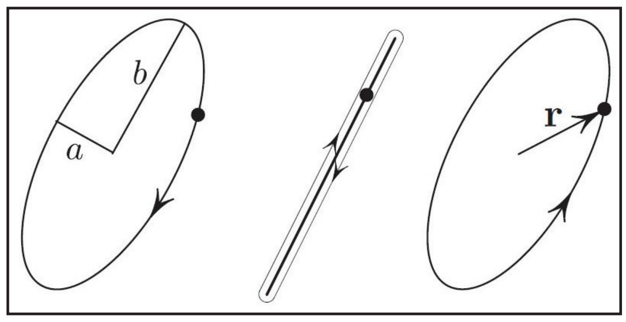

Figure 2. *Elliptical trajectories of (1) with negative, zero, and positive angular momentum respectively.*

Here are two explanations of this phenomenon, from two different angles.

Explanation 1. This explanation is based on the observation that *any solution of* $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$ *is a combination of two circular motions*, one counterclockwise and the other clockwise. In complex notation $\mathbf{r} = x + iy$, the general solution is

$$\mathbf{r} = Ae^{i\omega t} + Be^{-i\omega t},$$

where A and B are arbitrary complex constants. It follows that the solution in the rotating frame is

$$A + Be^{-2i\omega t},$$

as claimed.

Explanation 2. In addition to the restoring force, the bob in the rotating frame perceives two additional inertial (fictitious) forces acting on the bob: the centrifugal force $\omega^2\mathbf{R}$, where $\mathbf{R}$ is the bob's position expressed in the rotating frame, and the Coriolis force $-2i\omega\dot{\mathbf{R}}$. The apparent acceleration is thus the sum of the two inertial forces and the forces of the spring:

$$\ddot{\mathbf{R}} = -\omega^2\mathbf{R} + \omega^2\mathbf{R} - 2i\omega\dot{\mathbf{R}} = -2i\omega\dot{\mathbf{R}} \qquad (2)$$

(formally, one obtains (2) by substituting $\mathbf{r} = e^{i\omega t}\mathbf{R}$ into $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$ and simplifying). Note that the centrifugal force cancels the restoring force!

According to (2), the particle in the rotating frame is subject to the force normal to its velocity, same as the Lorentz force on a charged particle in the magnetic field of magnitude 2ω and normal to the plane. This demonstrates that the trajectories are circles (just as the trajectories of a charged particle are in the constant magnetic field perpendicular to the plane of the particle's motion).

Interestingly, passage to the rotating frame replaces the Hookean force by the Coriolis force, as just indicated.

The above equivalence is reversible; we conclude that the particle in a constant magnetic field, viewed in an appropriately rotating frame, behaves exactly as the planar harmonic oscillator (1).

I end with a tongue-in-cheek application to the Foucault pendulum, mounted over the North Pole. Wishing to match the pendulum's frequency to the Earth's angular velocity, we choose the length L to satisfy

$$\sqrt{g\,/\,L} = \frac{2\pi}{24 \cdot 3{,}600};$$

this gives $L \approx 1{,}176$ miles. A Foucault pendulum of this length, mounted over the North Pole, will execute circular motions of the type shown in Figure 3, making one revolution in 12 hours — provided that we raise the suspension point by, say, 100 miles to take the bob out of the atmosphere and prevent viscous drag. On a more realistic note, to observe this effect on a carousel making one revolution in six seconds, the length of the pendulum must be around 10 meters.

A more detailed discussion of this problem can be found in [1].

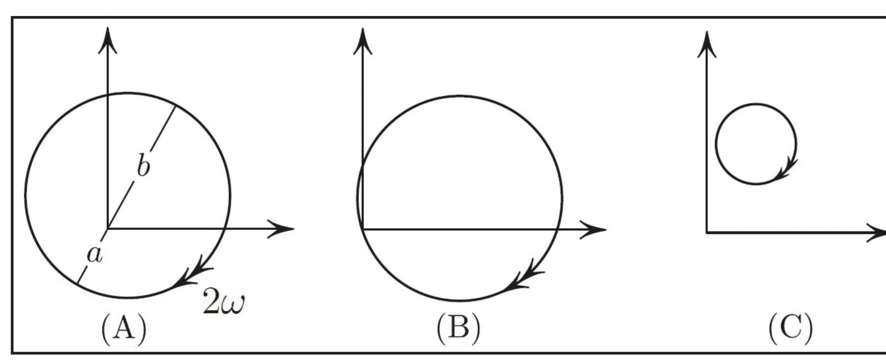

Figure 3. *Trajectories in the frame rotating counterclockwise with angular velocity ω, with negative (A), zero (B), and positive (C) angular momenta. The lengths a and b in (A) are the semiaxes of the ellipse in Figure 2.*

The figures in this article were provided by the author.

References

[1] Levi, M. (2014). *Classical Mechanics with Calculus of Variations and Optimal Control: an Intuitive Introduction*. In *Student Mathematical Library* (Book 69). Providence, RI: American Mathematical Society.

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