A Near-perfect Heat Exchange

MATHEMATICAL CURIOSITIES
By Mark Levi

When a hot and cold object are brought into contact with each other, their temperatures converge. And because temperatures have no inertia, they cannot "overshoot" and reverse. But nature found a remarkable way to overcome this constraint of the second law of thermodynamics and (nearly) exchange the two substances' temperatures (see Figure 1).

The Construction

The two liquids flowing in opposite directions are separated by a heat-conducting membrane. I claim that the temperatures of the two fluids will become nearly reversed: 0° water will heat to $100^\circ - \varepsilon$, while 100° water will cool to ε degrees, with an arbitrarily-small ε ($\varepsilon = 1^\circ$ in Figure 1).

How it Works

Figure 1 depicts a discrete approximation: we imagine each fluid split into n cells and replace the continuous motion with a jerky one. First we allow temperatures in adjacent cells across the membrane to settle to the common one (these will be slightly different in practice, contributing to ε in the preceding paragraph), and then we let each fluid quickly advance by one cell. This brings T_{k-1} and T_{k+1} into contact with one another. They settle to the common new temperature

$$T_k^+ = \frac{1}{2}(T_{k-1} + T_{k+1}) \qquad (1)$$

(the top will actually be slightly colder but we ignore this; we also ignore heat exchange between cells in the same pipe). In short, (1) is a discretization (in space and time) of the heat equation. Indeed, we can write it as

$$T_k^+ - T_k = \frac{1}{2}(T_{k-1} - 2T_k + T_{k+1}),$$
$$k = 1, \ldots n,$$

with $T_0 = 0$ (a new cell at 0° enters from the left) and $T_{n+1} = 100$ (a new cell at 100° enters from the right). The temperature profile will advance towards a linear one regardless of initial temperature distribution; the first cell will thus approach $T_1 = 100/(n+1)$ while the last cell will approach $T_n = 100 - 100/(n+1)$. A larger n means a more perfect temperature swap.

The key to the operation of the heat exchanger is that *the temperature differences are small in every heat exchange*: $T_{k+1} \approx T_{k-1}$. This proximity of temperatures makes for a small entropy increase. When heat Q passes from an object at temperature T_a to an object at temperature $T_b < T_a$, the entropy of the system consisting of these two objects increases by

$$Q\Big(\frac{1}{T_b} - \frac{1}{T_a}\Big) = Q\frac{T_a - T_b}{T_a T_b},$$

a small amount if $T_a \approx T_b$, even in Q is not small (here, T is the absolute temperature, *not* the centigrade). Since $T_{k+1} \approx T_{k-1}$, the heat exchanger increases entropy by less than an unintelligent design would. Speaking of intelligent design, biological evolution showed intelligence in "inventing" the heat exchanger. For example, deep veins in our arms run along the arteries, just like the two tubes in Figure 1 — the top tube represents a vein and the bottom one represents an artery. In cold weather, the outgoing arterial blood warms the cold venous blood coming from the hands; this helps maintain core body temperature.

The entropy's near-constancy signals the near-reversibility of the process [1]. This near-reversibility is also directly apparent from our ability to run Figure 1's outputs through another exchanger and nearly return to the original temperatures (e.g., to 2° and 98°).

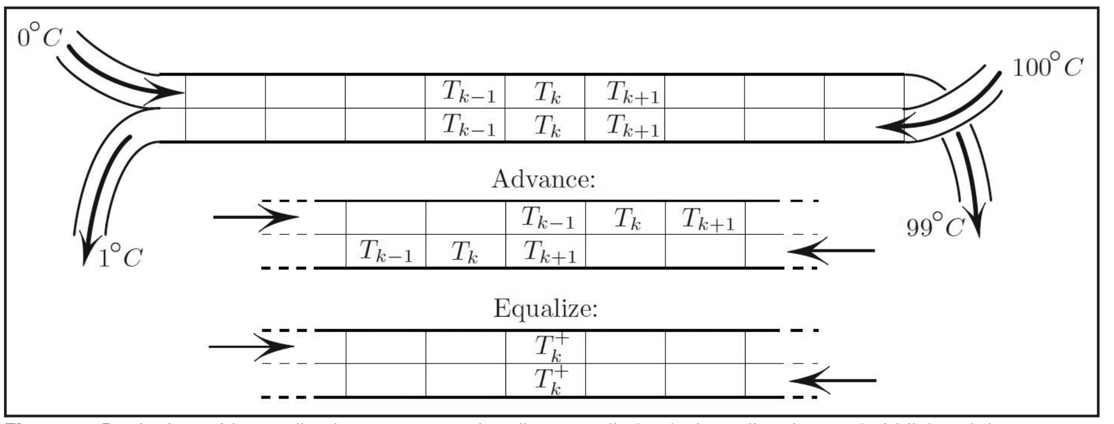

Figure 1. *Beginning with equalized temperatures in adjacent cells (top), the cells advance (middle) and the temperatures of adjacent cells (nearly) equalize. This completes the cycle. Figure courtesy of Mark Levi.*

References

[1] Feynman, R., Leighton, R., & Sands, M. (1964). Entropy. In *The Feynman Lectures on Physics* (Vol. 1). The California Institute of Technology.

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