Lagrange Multiplier as Depth or Pressure

MATHEMATICAL CURIOSITIES

By Mark Levi

With bicycle season beginning in the Northeastern U.S., I would like to describe a small bike-related observation. It actually has nothing to do with a bicycle's mechanics; it simply occurred to me when I was riding my bike last fall. While climbing up a steep incline and looking at a stream by the roadside, I asked myself the following question: *The water in any collection of connected vessels settles to the state of least potential energy; what is a mathematical expression of this obvious fact?* For cylindrical vessels it turned out to be the Cauchy-Schwarz inequality, as described in the November 2019 issue of *SIAM News*[1] (and in [2]). For polynomially tapered vessels, the expression yields Hölder's inequality [3].

These inequalities are therefore special cases of what every child knows: water levels equalize in communicating vessels. What other theorems are hiding behind this simple fact? In this column I provide one simple consequence; it would be interesting to discover more.

[1] https://sinews.siam.org/Details-Page/the-cauchy-schwarz-inequality-and-a-paradoxpuzzle

Incidentally, all of this—the Cauchy-Schwarz and Hölder inequalities, as well as the observation below—are, in the final analysis, *consequences of the law of conservation of energy*. Indeed, assume for a moment that water in communicating vessels settles at different levels. Then build a trough from the higher level to the lower one; the water will flow down this trough, and forever so due to the assumption, providing a free source of energy — a contradiction proving that the water settles at the same level, and also that this level minimizes potential energy.

As an aside, quite a few other geometrical theorems result from the impossibility of the perpetual motion machine [1].

Problem 1

Here is another problem that can be solved by the communicating vessels idea.

Given n *functions* $f_k:\mathbb{R}_+\to\mathbb{R}_+$, $k=1,\ldots n$, *minimize* the sum

$$F(x_1,\ldots,x_n)=\sum_{k=1}^{n}\int_0^{x_k} x f_k(x)dx, \qquad (1)$$

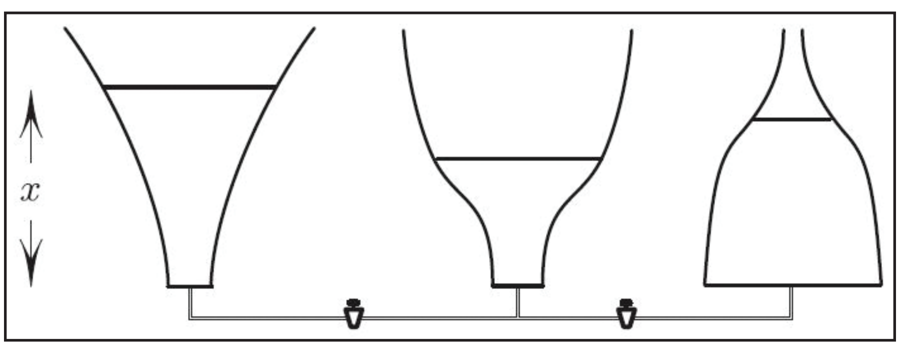

Figure 1. *$f_k(x)$ is the area of the horizontal cross-section at height x of the kth vessel. Figure courtesy of Mark Levi.*

subject to the constraint

$$G(x_1,\ldots,x_n)=\sum_{k=1}^{n}\int_0^{x_k} f_k(x)dx=1. \qquad (2)$$

To interpret this problem[2] physically, imagine n vessels (as in Figure 1) with valves closed and the kth vessel filled with water of depth x_k. The sum (1) is thus the system's total potential energy (we choose the units in which the water density and gravitational accelerations are one unit). And (2) prescribes the total volume of water. Now as we open the valves in Figure 1, the potential energy F settles to its least value, which also corresponds to equal levels:

$$x_k=x_l \quad 1\le k,l\le n. \qquad (3)$$

The total volume G remains unchanged during the redistribution. This solves the problem: the minimizer is given by (3) and (2).

To verify the answer, the Lagrange multipliers method $\nabla F=\lambda\nabla G$ yields

$$x_k \cancel{f(x_k)}=\lambda \cancel{f(x_k)},$$

so that $x_k=\lambda$ for all $k=1,\ldots,n$. As we already know, the levels of water equalize. But we now discover that *the Lagrange multiplier λ is the common water level, or equivalently, the water pressure at the bottom of the vessels.*

[2] To be more precise, we must assume that f_k are such that the constraint (2) is even satisfiable. I also probably should have said in fine print that $f_k\in L^1$, but I'll leave out these distracting details.

As a side remark, this problem generates the aforementioned inequalities for special choices of f_k.

Problem 2 (A Generalization)

Let $p_k:\mathbb{R}_+\to\mathbb{R}_+$, $1\le k\le n$ be monotone increasing functions, and let f_k be as it was before. Minimize

$$F_1(\mathbf{x})=\sum_{k=1}^{n}\int_0^{x_k} p_k(x)f_k(x)dx,$$

subject to the same previous constraint (2). The Lagrange multiplier method $\nabla F_1=\lambda\nabla G$ produces

$$p_k(x_k)=\lambda.$$

I leave it as a puzzle to build a thought-experimental "analog computer" that results in this answer and gives a physical interpretation of λ.

References

[1] Levi, M. (2009). *The mathematical mechanic: Using physical reasoning to solve problems*. Princeton, NJ: Princeton University Press.

[2] Levi, M. (2020). A water-based proof of the Cauchy-Schwarz Inequality. *Am. Math. Month.*, in press.

[3] Levi, M., & Tokieda, T. (2020). A communicating-vessels proof of Hölder's inequality. *Am. Math Month.*, in press.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.