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Geodesics on vibrating surfaces and curvature of the normal family

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Abstract
In this note we study the motion of a particle confined to a moving surface, in other words, the geodesic motion where the surface is allowed to vary. We show that in the case of a rapidly vibrating surface, the differential geometry of a certain family of normal curves plays a role. Certain curvature terms appear in the averaged equations of motion.

Mathematics Subject Classification: 70K70, 53Z05, 53C22

1. Introduction

Consider the motion of a particle confined to a moving surface

$$\varphi(x, y, z) - s(t) = 0, \tag{1}$$

where $s = \varepsilon s_1(t/\varepsilon)$, $s_1(\tau + 1) = s_1(\tau)$ is a periodic function and $\varepsilon > 0$. We assume that no forces, apart from the surface constraints, act on the particle. In other words, we study the classical problem of geodesic motion on surfaces, but with an added 'twist': the surface vibrates. We would like to understand the averaged effect of this vibration.

The main observation of this note is that a small-amplitude rapid vibration of the surface results in effective forces which can be explained geometrically in terms of curvature of the family of curves normal to the family of surfaces (1). The result of this note is a generalization to higher dimension of the observation in [14].

The effect described in this note is related to other remarkable phenomena that have been known in mechanics for some time:

(1) *Stabilization of the inverted pendulum by vertical vibration of the pivot point.* This problem deals with the motion of a point mass confined to a vibrating circle and is thus a one-dimensional version of the problem considered in the present note. There have been numerous papers on the vibrated inverted pendulum ([4, 8–14]); the phenomenon is described in several texts, including [3] and [7] . The underlying geometry of the problem has been pointed out [15].

(2) *Stabilization of a multiple pendulum.* There have been experiments showing that multiple pendula can be stabilized by vertical vibrations of the pivot [1, 2]. The (say) double pendulum with a vibrating pivot can be viewed as a point mass confined to a vibrating 2-torus in $\mathbb{R}^4$. The problem studied in this paper is different in only one respect: the surface we consider has codimension one.
(3) *The Paul trap.* A charged particle cannot be suspended stably in an electrostatic and gravitational field, since the potential is a harmonic function and thus has no minima away from the domain's boundary. Remarkably, if the voltage on the electrodes is made to oscillate fast enough, the equilibrium becomes stable; this is the principle of the Paul trap [18, 21] for which Paul was awarded a Nobel Prize. The stabilization mechanism of the Paul trap was given a geometrical explanation in [14].
(4) *Stabilization of the inverted fluid.* An experiment showing that viscous fluid can become stable in an upside-down vertically vibrating container has been reported in [17] and studied in [22]. Like the double pendulum, this system can be viewed as the motion of a particle confined to a vibrating surface. In this case the 'particle' is a diffeomorphism describing the position of all points in the fluid, while the 'surface' is a subset of volume-preserving diffeomorphisms in a set of diffeomorphisms.

The paper consists of two main parts. The first part is a very short heuristic derivation of the averaged equations of motion; this derivation, in our opinion, represents the main interest, since it uncovers in one short paragraph the geometry which remains hidden in a long analytic calculation. The second part of the paper serves to prove that the heuristic guess is correct. It is a testimony to the power of heuristics that one can write the averaged equations of motion without even writing the original equations, and that the heuristic derivation is shorter than the rigorous proof by a large factor, and that, in addition, the heuristic derivation results in a more elegant (but fortunately equivalent) form of the averaged equation. Unfortunately, it is not clear to us how to make the heuristic argument rigorous. To prove the heuristic result we are forced to take a different path of using transformation theory.

The second part of the paper (sections 3–5) is organized as follows. We derive the equations of motion (section 3), to which we apply a general averaging theorem (section 4). The resulting equations do not look like the heuristically derived equations, since the latter include some geometrical terms like the curvature. We prove that the two equations are actually the same in section 5.

Remark 1. While the class of time-dependent surfaces (1) allows deformation as well as vibration, it does not include a more general class $\varphi(x, y, z, t) = 0$. A simple example not covered by equation (1) is

$$z - \varepsilon \cos(x - t/\varepsilon) = 0. \tag{2}$$

No new phenomena arise in this more general case. However, in the presence of dissipation—a force linear in the velocity—there arises an additional 'geometrical phase' effect (see [16] for the treatment in $\mathbb{R}^2$). This effect occurs when the vibration of the surface is non-reversible, as, for instance, in the case of equation (2). We do not address this question in this note.

2. A heuristic derivation of averaged equations

When ε is small, we expect the motion to consist of the 'averaged' part $\boldsymbol{R}$ and the superimposed fluctuations:

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{h}(\boldsymbol{R}, t) \tag{3}$$

The following theorem describes the motion of the 'guiding centre' $\boldsymbol{R}$.

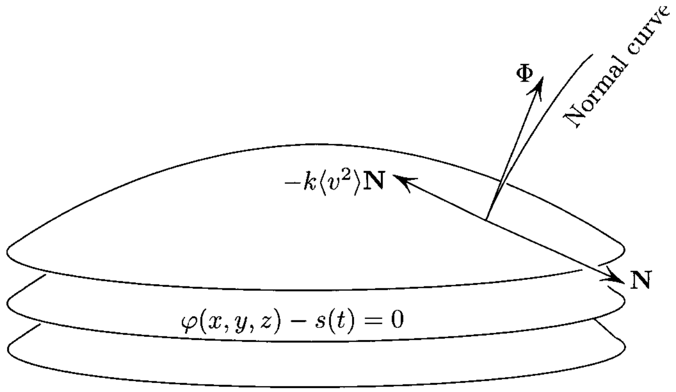

Figure 1. The effective force $k\langle v^2\rangle N$ due to the vibration of the surface.

Theorem 1. *Consider the family of surfaces (1) parametrized by t. This family generates a family of normal curves in $\mathbb{R}^3$, if the gradient $\nabla\varphi = \varphi' \neq 0$ and $\dot{s} \neq 0$. Let $k = k(\boldsymbol{R})$ be the curvature of the normal curve passing through $\boldsymbol{R} \in \mathbb{R}^3$, and let $\boldsymbol{N} = \boldsymbol{N}(\boldsymbol{R})$ be the principal normal vector to this curve at $\boldsymbol{R}$. Finally, let v be the normal velocity (with t) of the moving surface (1). It is easy to see that $v = \dot{s}/|\varphi'|$. We claim that there exists a transformation (3) which decomposes the motion into the average motion $\boldsymbol{R}$ and the fluctuation $\boldsymbol{h} = O(\varepsilon)$ such that $\boldsymbol{R}$ satisfies*

$$\ddot{\boldsymbol{R}} = -(\varphi''\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}})\boldsymbol{\Phi} - k\langle v^2\rangle \boldsymbol{N} + \boldsymbol{E}, \tag{4}$$

where $\boldsymbol{E}$ is small with ε: $\boldsymbol{E} = o(\varepsilon^0)$ and $\boldsymbol{\Phi}$ is the rescaled gradient

$$\boldsymbol{\Phi} = \frac{\varphi'}{|\varphi'|^2}, \qquad \text{where } \varphi' = \nabla\varphi. \tag{5}$$

Remark 2. The truncated averaged equation

$$\ddot{\boldsymbol{R}} = -(\varphi''\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}})\boldsymbol{\Phi} - k\langle v^2\rangle \boldsymbol{N} \tag{6}$$

leaves the tangent bundle of the surface $S := \{\varphi = \text{const}\}$ invariant: for any solution $\boldsymbol{R}(t)$ of equation (6), if $\boldsymbol{R}(t) \in S$ and $\dot{\boldsymbol{R}}(t) \in T_{\boldsymbol{R}(t)}S$ for some t, then the same is true for all t.

To prove the remark we start with the particle confined to a fixed surface S and subjected to an additional force $-k\langle v^2\rangle \boldsymbol{N}$ which, we observe, is tangent to S. Following the steps of the derivation of equation (9) for the time-independent surface (see section 3 below), we obtain equation (6).

Heuristic derivation of equation (4). The rigorous proof is contained in sections 3–5. Since $s(t) = \varepsilon s_1(t/\varepsilon)$ is rapidly oscillating, we expect the particle to move primarily in the normal direction to the surface, i.e. along one of the curves from the normal family. Had the particle been actually restricted to a normal curve, the particle would have felt the centripetal force $kv^2\boldsymbol{N}$ (figure 1). The average of this force over the period ε is $k\langle v^2\rangle \boldsymbol{N}$. Here $\langle f(\tau)\rangle$ denotes the average of f over $0 \leqslant t \leqslant \varepsilon$.

Since in reality the constraint does not exist, the particle should behave as if it felt the force $-k\langle v^2\rangle \boldsymbol{N}$. In addition to this force, the particle also feels the centripetal force

$$-(\varphi''\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}})\boldsymbol{\Phi}$$

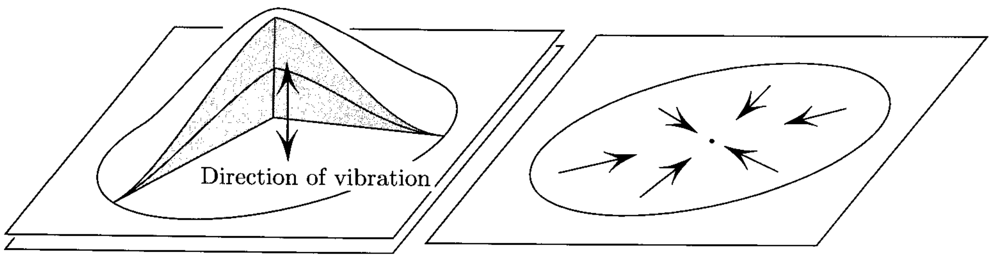

Figure 2. The vibration of a surface patch creates an effective potential force attracting towards the point of maximal amplitude.

(This figure is in colour only in the electronic version)

due to the constraint to the surface[1]. Actually, there is an additional D'Alembert force which comes from the surface's acceleration, but that force averages to zero over the period since $s(t)$ is periodic.

This is the heuristic explanation of our claim (3). In the following sections we justify this approach. To that end we derive the equations of motion (section 3) and apply the averaging technique (section 4). The averaged equations obtained by a formal procedure look quite different from the geometrical form equation (4); we will show that in fact the two forms are equivalent (section 5).

It takes several pages of technical effort to prove validity of the heuristic idea which takes only a few sentences to express. This in a way shows the power of heuristic reasoning.

An example. This example illustrates how vibration creates an effective force attracting towards the area of greatest amplitude. Choosing $\varphi(x, y, z) = z\,e^{x^2+y^2}$, $s(t) = \varepsilon \sin(t/\varepsilon)$, we obtain a vibrating Gaussian bump

$$z = \varepsilon \sin(t/\varepsilon) e^{-x^2-y^2}. \tag{7}$$

The first term on the right-hand side in the averaged equation (4) vanishes on the invariant plane $z = 0$, while $\boldsymbol{N} = -|\boldsymbol{R}|^{-1}\boldsymbol{R}$, and equation (4) reduces to

$$\ddot{\boldsymbol{R}} = -k\langle v^2\rangle |\boldsymbol{R}|^{-1}\boldsymbol{R} + \boldsymbol{E}. \tag{8}$$

This form shows, in particular, that there is an effective attractive force towards the origin. The vibrating bump creates a potential well (figure 2).

3. Equations of motion

In this section, we derive the equation of motion for a particle constrained to a time-dependent surface.

Lemma 1. *The motion of a point mass confined to the time-varying surface* $\varphi(\boldsymbol{x}) - s(t) = 0$ *with no forces besides the constraint is governed by*

$$\ddot{\boldsymbol{x}} = \boldsymbol{F} + \ddot{s}\boldsymbol{\Phi}, \tag{9}$$

where

$$\boldsymbol{F} = -(\varphi''\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{\Phi} \quad \text{and} \quad \boldsymbol{\Phi} = \frac{\varphi'}{|\varphi'|^2}. \tag{10}$$

[1] This expression is derived in section 3.

Note that $\boldsymbol{F}$ is the centrifugal force (for stationary surface) and $\ddot{s}\boldsymbol{\Phi}$ is the D'Alembert force (the additional force upon the particle due to the surface's acceleration).

Proof. Differentiating $\varphi(\boldsymbol{x}) - s(t) = 0$ by t, we have $\varphi' \cdot \dot{\boldsymbol{x}} - \dot{s} = 0$. Another t-differentiation gives

$$(\varphi''\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \varphi'\ddot{\boldsymbol{x}} - \ddot{s} = 0. \tag{11}$$

Since there are no frictional forces, we have $\ddot{\boldsymbol{x}} = \lambda\varphi'$ for some real λ, and it remains to find the Lagrange multiplier λ. Substituting the last expression of $\ddot{\boldsymbol{x}}$ into equation (11), we obtain

$$(\varphi''\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + |\varphi'|^2\lambda - \ddot{s} = 0,$$

which gives

$$\lambda = -\frac{(\varphi''\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) - \ddot{s}}{|\varphi'|^2}$$

and substituting this value of λ into $\ddot{\boldsymbol{x}} = \lambda\varphi'$ results in equation (9). □

4. Averaging the equations of motion

In this section we average equation (9) and then verify that the averaged equation indeed coincides with the heuristically guessed equation (4). One can prove, using the averaging procedure (see, e.g. [5, 6, 15, 19, 20]), the following.

Theorem 2. *There exists a transformation*

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{h}(\boldsymbol{R}, t, \varepsilon) + \cdots$$

such that equation (9) transforms into

$$\ddot{\boldsymbol{R}} = \boldsymbol{F}(\boldsymbol{R}, \dot{\boldsymbol{R}}) + \tfrac{1}{2}(F_{\dot{R}\dot{R}}\boldsymbol{\Phi}, \boldsymbol{\Phi}) - \langle\dot{s}^2\rangle\boldsymbol{\Phi}'\boldsymbol{\Phi} + \boldsymbol{E},$$

where $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ are defined in (10), and where $\boldsymbol{E} = o(\varepsilon^0)$—it should be noted that the preceding term is of order one: $\dot{s} = O(\varepsilon^0)$, so that $\boldsymbol{E}$ indeed is a higher order term.

Substituting the expressions for $\boldsymbol{F}$ and $\boldsymbol{\Phi}$, we obtain the averaged equation

$$\ddot{\boldsymbol{R}} = -(\varphi''\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}})\boldsymbol{\Phi} - \langle\dot{s}^2\rangle((\varphi''\boldsymbol{\Phi}, \boldsymbol{\Phi})\boldsymbol{\Phi} + \boldsymbol{\Phi}'\boldsymbol{\Phi}) + \boldsymbol{E}. \tag{12}$$

5. Curvature of the normal family and the equivalence of equations (4) and (12)

To prove the equivalence of the two equations, it suffices to prove the identity (13) below.

Lemma 2. *Consider the family $\mathcal{F}$ of surfaces of the form $\{\boldsymbol{x} : \varphi(\boldsymbol{x}) = s(t)\}$, $t \in \mathbb{R}$, and let $\mathcal{F}^\perp$ be the family of curves normal to the family $\mathcal{F}$, see figure 1. Let k be the principal curvature and let $\boldsymbol{N}$ be the principal normal vector of a curve from $\mathcal{F}^\perp$. Then*

$$k\langle v^2\rangle\boldsymbol{N} = \langle\dot{s}^2\rangle((\varphi''\boldsymbol{\Phi}, \boldsymbol{\Phi})\boldsymbol{\Phi} + \boldsymbol{\Phi}'\boldsymbol{\Phi}). \tag{13}$$

Proof. Let $\boldsymbol{r}(\sigma)$ be a curve from $\mathcal{F}^\perp$, parametrized by the arclength σ; we have $(\mathrm{d}/\mathrm{d}\sigma)\boldsymbol{r} = \varphi'/|\varphi'|$. We have

$$k\langle v^2\rangle\boldsymbol{N} = \langle v^2\rangle\boldsymbol{r}'' = \langle v^2\rangle\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\varphi'}{|\varphi'|}\right) = \langle v^2\rangle|\varphi'|^{-2}\left(\frac{\mathrm{d}\varphi'}{\mathrm{d}\sigma}|\varphi'| - \varphi'\frac{\mathrm{d}|\varphi'|}{\mathrm{d}\sigma}\right). \tag{14}$$

Now,

$$\frac{\mathrm{d}\varphi'}{\mathrm{d}\sigma} = \varphi''\boldsymbol{r}' = \frac{\varphi''\varphi'}{|\varphi'|} \tag{15}$$

and

$$\frac{\mathrm{d}|\varphi'|}{\mathrm{d}\sigma} = \frac{\mathrm{d}}{\mathrm{d}\sigma}(\varphi', \varphi')^{1/2} = \frac{(\varphi''\varphi', \varphi')}{|\varphi'|^2}. \tag{16}$$

Also, since $v = \dot{s}/|\varphi'|$, we have

$$\langle v^2 \rangle = \frac{\langle \dot{s}^2 \rangle}{|\varphi'|^2}. \tag{17}$$

Substituting equations (15)–(17) into equation (14) we obtain

$$\begin{aligned} k\langle v^2\rangle \boldsymbol{N} &= \langle \dot{s}^2\rangle \frac{|\varphi'|^2\varphi''\varphi' - (\varphi''\varphi', \varphi')\varphi'}{|\varphi'|^6}, \\ &= \langle \dot{s}^2\rangle \frac{(\varphi''\varphi', \varphi')\varphi' + (|\varphi'|^2\varphi''\varphi' - 2(\varphi''\varphi', \varphi')\varphi')}{|\varphi'|^6}, \\ &= \langle \dot{s}^2\rangle \left(\frac{(\varphi''\varphi', \varphi')\varphi'}{|\varphi'|^6} + \left(\frac{\varphi'}{|\varphi'|^2} \right)' \frac{\varphi'}{|\varphi'|^2} \right). \end{aligned}$$

Recalling that $\varphi'/|\varphi'|^2 = \boldsymbol{\Phi}$, we obtain equation (13). The proof of the lemma, and thus of the equivalence of the two equations, and thus of theorem 1 is complete. □

6. Conclusions

We have explored the close connection between averaging and geometry in rapidly vibrating systems. For a particle on a vibrating surface, we introduce an auxiliary problem with 'normal constraints'. More precisely, in addition to constraining the particle to a vibrating surface, we artificially constrain it to a curve from the normal family. We then focus our attention on the force of this artificially imposed constraint. The main conclusion of this note: vibration of the surface gives rise to a new effective force *equal to the negative of the force of the artificial constraint.*

This observation makes it geometrically clear why the particles confined to vibrating surfaces are attracted to the area of the greatest amplitude.

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References

[1] Acheson D J 1993 A pendulum theorem *Proc. R. Soc. Lond.* A **443** 239–45
[2] Acheson D J and Mullin T 1993 Upside-down pendulums *Nature* **366** 215–16
[3] Arnold V I 1992 *Ordinary Differential Equations* (Berlin: Springer)
[4] Blitzer L 1965 Inverted pendulum *Am. J. Phys.* **33** 1076
[5] Bogoliubov N N and Mitropolskii Y A 1961 *Asymptotic Methods in the Theory of Non-linear oscillations* (New York: Gordon and Breach)
[6] Bullo F 2002 Averaging and vibrational control of mechanical systems *SIAM J. Control Optim.* **41** 542–62
[7] Feynman R 1965 *The Feynman Lectures on Physics* (Reading, MA: Addison-Wesley)
[8] Kalmus H P 1970 The inverted pendulum *Am. J. Phys.* **38** 874

[9] Kaptisa P L 1965 Dynamical stability of a pendulum when its point of suspension vibrates *Collected Papers by P L Kapitsa* vol II (London: Pergamon) pp 714–25
[10] Kaptisa P L 1965 Pendulum with a vibrating suspension *Collected Papers by P L Kapitsa* vol II (London: Pergamon) pp 726–37
[11] King R E 1965 Inverted pendulum *Am. J. Phys.* **33** 855
[12] Landau L D and Lifshitz E M 1976 *Course of Theoretical Physics* vol 1 (Oxford: Pergamon)
[13] Levi M 1988 Stability of the inverted pendulum—a topological explanation *SIAM Rev.* **30** 639–44
[14] Levi M 1998 Geometry of Kapitsa's potentials *Nonlinearity* **11** 1365–8
[15] Levi M 1999 Geometry and physics of averaging with applications *Physica* D **132** 150–64
[16] Levi M and Weckesser W 1995 Stabilization of the inverted linearized pendulum by vibration *SIAM Rev.* **37** 219–23
[17] Michaelis M M and Woodward T 1991 An inverted liquid demonstration *Am. J. Phys.* **59** 816–21
[18] Paul W 1990 Electromagnetic traps for charged and neutral particles *Rev. Mod. Phys.* **62** 531–40
[19] Bellman R, Bentsman J and Meerkov S M 1986 Vibrational control of nonlinear systems: vibrational stabilizability *IEEE Trans. Automat. Control.* **31** 710–24
[20] Baillieul J, Weibel S and Kaper T 1997 Global dynamics of a rapidly forced cart and pendulum *Nonlinear Dyn.* **13** 131–70
[21] Osberghaus O, Paul W and Fischer E 1958 Electromagnetic traps for charged and neutral particles. *Forschungsberichte des Wirschafts- und Verkersministeriums Nordrhein-Westfalen* vol 415 (Koln and Opladen: Westdeutscher)
[22] Wolf G H 1970 Dynamic stabilization of the interchange instability of a liquid–gas interface *Phys. Rev. Lett.* **24** 444–6