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# Morse theory for a model space structure.

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In this note we outline the results of qualitative analysis of a natural perturbation of the classical problem of dynamics of a free rigid body in space. Our aim here is to describe a bifurcation effect previously unobserved in flexible space structures. To that end we choose a maximally simple model to make the ideas as transparent as possible with the minimum of technicalities.

By coupling two classical and well-understood problems–a free rigid body in space and a damped oscillator we produce a dynamical system which can be viewed as the simplest meaningful model of a flexible body, such as a spacecraft with flexible attachments. In fact, our system can be viewed as a **one-mode truncation** of a flexible structure with infinitely many degrees of freedom. The phenomena described here are certain to occur in such infinite degree of freedom structures as well.

One such surprising phenomenon is the "symmetry–seeking" property of the model, manifesting itself in the existence of purely rotational asymptotic motions in which the effective ellipsoid of inertia of the rotating body becomes rotationally symmetric after the internal vibrations have dissipated, for a whole **interval** of angular momentum values. Moreover, the axis of rotation of the system will not coincide in these motions with any of the symmetry axes of the system. Finally, the deflections of the elastic part of the system will be shown to be *independent of the value of the angular momentum* in a certain range.

It seems to be a common belief that any elastic structure with internal dissipation tends, in the absence of all external forces, to a pure rotation with a constantt angular velocity around one of the axes of the effective ellipsoid of inertia (cf. Kaplan [5]). We show that, generally speaking, this is false, and propose a condition under which the asymptotic motions indeed are pure rotations.

In the presence of internal dissipation the flow in the reduced state space the system is almost always a Morse flow, that is, every motion tends to an equilibrium, with the energy playing the role of the Lyapunov function. Physical implications of this transparent geometrical picture may seem rather surprising; these are described in detail below.

## 1. Introduction

The problems of dynamics of non-rigid structures in space have been studied extensively, notably, by Poincaré, Lyapunov, Kovalevskaya, and more recently by Chandrasekhar [4] and others. These remarkable studies deal with the equilibrium shapes of self-gravitating fluids.

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Since the early 1960's the new interest in related questions arose in connection with the development of artificial satellites, and more recently, in the context of large flexible space structures.

We mention in passing a very interesting problem of the motion of a rigid body with a fluid-filled cavity; for the ideal fluid with no bubbles, the problem has been solved completely over 100 years ago [8]. As it turns out, the motion of the fluid-filled body is indistinguishable from that of a rigid body with an appropriate tensor of inertia(!).

Recent ideas and methods of dynamical systems have not yet exercised their full power on this class of problems, and have only started penetrating this particular area. Very recently, various aspects of dynamics of flexible space structures have attracted attention of mathematicians. Among the new studies are stability proofs of the "least energy rotations" based on Arnold's ideas [2],[7], bifurcation analyses of rotating systems consisting of a rigid part with flexible attachments [3], [7], studies of the dynamics of coupled rigid bodies; the references to this subject can be found in the present volume.

The Hamiltonian approach of Arnold [2] has been carried over to some models [7] and the Lagrangian approach was developed [3].

We mention also a somewhat related work starting with Kolodner [1], [6], [9] on the equilibria and dynamics of moving chains.

Our understanding of the full dynamics of flexible space structures has been very limited. Almost all existing results on the qualitative aspects of dynamics fall into the following categories:

- The characterization of "rigid modes" in *some* models, i.e. the classification (in a very few special examples) of possible equilibrium motions and of their bifurcations [3].
- Proof (in a few examples) of stability of rigid rotations in the least energy case [7].
- Proof (in some special cases) of the trend to a pure rotational motion [3].

While analyzing the effects of infinite-dimensionality of the beam, the body–beam model [3] deliberately excluded the effects of spatial orientation. In this note, we give a complete description of a model problem possessing the full rotational freedom.

## 2. Description of the model

We consider a rigid body, Figure 2.1, whose tensor of inertia is $I$, and we orient the coordinate axes $x, y, z$ along the eigendirections of $I$, denoting the corresponding eigenvalues, i.e. the moments of inertia, by $I_1 > I_2 > I_3$. Let a mass $m$ be constrained to slide along the $x$–axis and let it be a subject to an elastic restoring force $k(x - x_0)$ with the spring constant $k$. Here $x$ denotes the position of the mass along the $x$–axis and $x_0$ is the equilibrium position of the mass when the entire system is at rest. To reduce the technicalities to a minimum, we also assume the center of mass of the rigid body to be fixed in space–or alternatively, we could take another mass $m_1 = m$ positioned symmetrically with the first mass, with symmetric initial conditions. Let $c\dot{x}$ be the damping force acting on the mass. We emphasize that the linearity assumptions on the elasticity and the dissipation have nothing to do with the phenomena described below; these assumptions are chosen to bring out the essential features unobstructed by inessential technicalities. The equations of motion of the system are given by

$$\begin{cases} \dot{M} + \omega \times M = 0, \\ \ddot{x} + c\dot{x} + k(x - x_0) = (\omega_2^2 + \omega_3^2)x, \end{cases} \tag{2.1}$$

where $M = I\omega + mr \times (r_t + \omega \times r)$ is the total angular momentum of the system expressed in the body frame, $\omega$ is the angular velocity of the rigid body expressed in the body frame and $r$ is the position vector of the mass in that frame. This form of equations follows at once from the general Lagrangian aproach [3]; one can also derive them directly. The first equation, for instance, expresses the conservation of the angular momentum but written in the non–inertial frame of the rigid body. Since the vectors $r$ and $r_t$ are parallel, we have $M = I\omega + mr \times (\omega \times r) \equiv I(x)\omega$, where $I(x) = diag(I_1, I_2 + x^2, I_3 + x^2)$ is the (variable) tensor of inertia of the whole system; we will refer to it as the effective tensor of inertia. The equations of can now be written as

$$\begin{cases} \frac{d}{dt}[I(x)\omega] + \omega \times I(x)\omega = 0 \\ \ddot{x} + c\dot{x} + k(x - x_0) = (\omega_2^2 + \omega_3^2)x, \end{cases} \tag{2.2}$$

where we took the mass $m = 1$, or more explicitly,

$$\begin{cases} I\dot{\omega} + \omega \times I\omega + 2x\dot{x}\omega^1 + x^2\dot{\omega}^1 + x^2\omega_1(0, -\omega_3, \omega_2)^T = 0 \\ \ddot{x} + c\dot{x} + k(x - x_0) = (\omega_2^2 + \omega_3^2)x, \end{cases} \tag{2.3}$$

where $\omega^1 = (0, \omega_2, \omega_3)^T$. Physical meaning of the terms in the first equation of (2.3) is as follows.

- — the third term gives the Coriolis torque exerted by the mass upon the rigid body.
- — the fourth term gives the torque created by the inertial force due to the angular acceleration of the rigid body
- — the last term gives the torque created by the centrifugal (inertial) force.

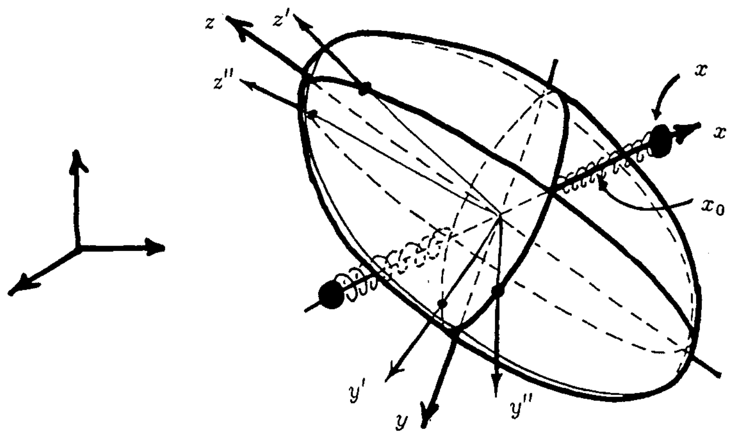


Figure 2.1. A model problem.

The right-hand side term in the second equation in (2.3) gives the centrifugal inertial force acting upon the mass exerted by the constraint of staying on the $x$-axis.

Energy of the system is given by

$$E = \frac{1}{2}(I\omega, \omega) + \frac{1}{2}[\dot{x}^2 + (\omega_2^2 + \omega_3^2)x^2] + \frac{1}{2}k(x - x_0)^2,$$

the expression in bracket giving the kinetic energy of the mass. The dissipation relation

$$\dot{E} = -c\dot{x}^2$$

can be checked either directly or from the general principle of Routh (cf. [3]). From the left-invariance of the Lagrangian, i.e. from the independence of the kinetic energy of the position of the body relative to the inertial frame, it follows that $|M|$ is a conserved quantity; in the presence of dissipation ($c \neq 0$) it is the only nontrivial conserved quantity.

## 3. Dynamics and Bifurcations

In this section we describe the asymptotic dynamics of the system and point out some interesting "symmetry-seeking" effects.

The first step in the direction was made in [3], although the first heuristic discussion is mentioned already in [5]. As it turns out, that heuristic discussion is valid only for moderate values of angular momentum.

### 3.1 Asymptotic behavior.

**Theorem 3.1.** *Every solution of eq. (2) tends to the zero dissipation set (the set of rigid configurations)*

$$\{(x, \dot{x}, \omega) : \dot{x} = 0\},$$

*if the damping* $c > 0$.

**Proof.** We have proved a similar theorem in [3] for an infinite- dimensional system but with only one rotational degree of freedom; the main difference here is that the system has a full rotational freedom.

We have to show that given any $\epsilon > 0$ and any initial condition $Z_0 = (x(0), \dot{x}(0), \omega_0)$, there exists $T = T(Z_0, \epsilon)$ such that for all $t \geq T$ the solution $Z(t, Z_0) = (x, \dot{x}, \omega)$ stays in the $\epsilon$-strip around a $(x, \omega)$-hyperplane in $\mathbf{R}^5 \equiv \{(x, \dot{x}, \omega)\}$. Assuming the contrary, we would have an $\epsilon > 0$ and $Z_0$ such that the solution starting at $Z_0$ would venture repeatedly outside the strip $|\dot{x}| \leq \epsilon$. For $|\dot{x}| > \epsilon$ we have $\dot{E} = -c\dot{x}^2 < -c\epsilon^2$, and thus we conclude that $Z(t)$ can spend only finite time outside the strip $|\dot{x}| \leq \epsilon$; since $\epsilon > 0$ was arbitrary, $x(t)$ will repeatedly come arbitrarily close to $\{\dot{x} = 0\}$. Consequently the strip $\frac{\epsilon}{2} \leq \dot{x} \leq \epsilon$ will be crossed infinitely many times. Since the amount of energy lost in each crossing is greater than some $\delta > 0$, we conclude: $E \to -\infty$, a contradiction.

We have used the same argument in an infinite-dimensional setting in [3]. The above result says that the pendulum stops oscillating, but leaves open the question on the limiting motion of the system as a whole. This question is answered in the next statement.

**Theorem 3.2** *If $I_2 \neq I_3$ then for any initial condition in eq. (2.2) we have $\omega(t) \to \omega_\infty = const$, $x(t) \to x_\infty = const$, $\dot{x} \to 0$, with the limiting values satisfying*

$$I(x_\infty)\omega_\infty = \lambda\omega_\infty, \quad x_\infty = \frac{kx_0}{k - \omega_2^2 - \omega_3^2}.$$

*We conclude that eq. (2.2) is a Morse system for $I_2 \neq I_3$.* **Proof.** The proof is the same as for a similar theorem in [3].

**Remark.** It is important to note that the apparently reasonable conclusion $\omega \to const$ is false in general: if $I_2 = I_3$, a "typical" solution tends to $x \to const$, $\omega \to (\omega_1, a\cos b(t-t_0), a\sin b(t-t_0))$ with constant $a, b, \omega_1$. Physically, this says that even in the presence of elasticity some structures need not tend to a pure rotation but rather may tend asymptotically to a precessing motion. One expects, however, that this is an exceptional phenomenon; in the case $I_2 = I_3$ at hand it is – in fact, the asymptotic precession is nongeneric and in fact, is of codimension 1, as theorem 3.2 shows. It should be pointed out that this phenomenon is due to a "residual rigidity" in the system: the motion of the mass is constrained to a line. Were we to relax this constraint (thus also increasing the dimension of the problem), we would have eliminated the possibility of a limiting motion with precession. It should also be pointed out that in the presence of the symmetry $I_2 = I_3$ one can carry out a further reduction; in the reduced phase space the system tends to an equilibrium and is a Morse system (which it is not in $\mathcal{M}_\mu$ in the symmetric case).

The model at hand suggests that, as a general principle, in order to eliminate the possibility of an asymptotic precession in a flexible space structure it suffices to assume that any change in the angular velocity of the structure leads to a deformation of some flexible part of the structure. To illustrate this point on the present model we note that the last condition fails precisely in the case $I_2 - I_3$; indeed, for a precession we have $\dot{\omega} = (0, -ab\sin bt, ab\cos bt) \neq 0$, which produces no change in the inertial forces acting upon the mass and thus no deformation; had the constraint of the mass to the $x$–axis been relaxed, the precessional motion would not have existed as an asymptotic state.

### 3.2 Global dynamics

Theorem 3.2 above describes the fate of an individual solution, but leaves open the question of global dynamics of the system and of its bifurcations – that is, it does not say what happens as one changes the initial consitions or the parameters. In this section we describe the effect by which the system "seeks out" a symmetry which persists for an open interval of parameter values.

Before stating the result, we note that the phase space $R^5$ is foliated by the invariant surfaces of constant angular momentum, and it suffices therefore to analyze the flow of eq. (2) restricted to the angular momentum surface

$$\mathcal{M}_\mu = \{(\omega, x, \dot{x}) : |M| = |I(x)\omega| = \mu\},$$

for different values of the constant $\mu$. Topologically, $M_\mu = S^2 \times \mathbf{R}^2$. We fix the magnitude $\mu$ of the angular momentum and choose the Hooke's constant $k$ of the spring as the parameter (the choice of $\mu$ as the parameter instead of $k$ would amount simply to a rescaling of time).

**Theorem 3.3** *Fix $|M| = \mu$, and assume that $I_1 > I_2 + x_0^2$, i.e. that the effective principal moments of inertia $I_1 > I_2 + x_0^2 > I_3 + x_0^2$ of the undeformed system are ordered in the same way as the moments of inertia $I_1 > I_2 > I_3$ of the rigid body. For any $k$ in the interval*

$$0 < k < \frac{\mu}{I_1^2}\left(1 - \frac{x_0}{\sqrt{I_1 - I_2}}\right)^{-1} \equiv k_y \tag{3.1}$$

*the system admits a rotation around two axes $y', y''$ in the $(x, y)$-plane not coinciding with any of the principal inertial axes $x$ or $y$. Moreover, the position $x_\infty$ of the mass in such a limiting motion is independent of $k$. This position is such as to make the effective ellipsoid of inertia rotaionally symmetric around the $z$-axis. The same statements hold when one replaces the $y$-axis with the $z$-axis, the $xy$-plane with the $xz$-plane, $I_2$ with $I_3$, $k_y$ with $k_z = \frac{\mu}{I_1^2}(1 - \frac{x_0}{\sqrt{I_1 - I_3}})^{-1}$, etc..*

**Proof.** We start by defining $x_\infty$ by the symmetry condition $I_1 = I_2(x_\infty) \equiv I_2 + x_\infty^2$, i.e. by

$$x_\infty^2 = I_1 - I_2 > x_0^2.$$

Next we define the angular velocity $\omega_\infty$; in order that eq. (2.2b) hold for the constant $x(t) \equiv x_\infty$, it is necessary that

$$\omega_2^2 + \omega_3^2 = k\frac{x_\infty - x_0}{x_\infty} > 0;$$

the last inequality follows from the assumption made in the statement of the theorem. For eq. (2.2a) to hold one must have $\omega \times I(x_\infty)\omega = 0$, i.e. there must exist $\lambda > 0$ such that $I(x_\infty)\omega = \lambda\omega$, or $(I_1\omega_1, I_1\omega_2, (I_3 + x_\infty^2)\omega_3) = \lambda(\omega_1, \omega_2, \omega_3)$. Choosing $\omega_3 = 0$, $\lambda = I_1$, we satisfy eq. (2.2a); eq. (2.2) is thus satisfied by any $\omega = (\omega_1, \omega_2, 0)$ with

$$\omega_2^2 = k\frac{x_\infty - x_0}{x_\infty} > 0 \quad \text{and} \quad x = x_\infty \equiv I_1 - I_2. \tag{3.2}$$

Recalling the angular momentum constraint $|M| = \mu$, we obtain $\omega_1^2 + \omega_2^2 = \frac{\mu}{2I_1^2}$. Thus if

$$A \equiv \frac{\mu}{I_1^2} - k\frac{x_\infty - x_0}{x_\infty} > 0 \tag{3.3}$$

then $\omega_1 = \sqrt{A}$ and $\omega_2$ given by eq. (3.2) satisfy the equations of motion (2.2). Condition (3.3) is in turn equivalent to (3.1), q.e.d. .

### 3.3. Bifurcations and stability.

It is not hard to show that for certain values of the parameters our system has **seven** distinct equilibrium rotations, among which three can be stable; all these are depicted in figure 3.1. (We identify, of course, the rotations around the same axis in the opposite direction.) This is to be contrasted with the well-known situation of only three rotations for the rigid bodies with distinct principal moments of inertia. For large $k$ the spring is stiff and the system has three equilibrium rotations only, just as a rigid body. As $k$ is decreased it passes through two bifurcation values $k_z > k_y > 0$ specified in Theorem 3.1. At $k_z$ the rotational equilibrium around the $z$-axis undergoes a pitchfork bifurcation giving rise to two additional rotations around the $z'$ and $z''$-axes which bifurcate off of the $z$-axis and which lie in the $xz$-plane. As $k$ decreases further, these axes rotate towards the $x$-axis as shown in figure 3.1. As $k$ decreases through the next value $k_y < k_z$, two axes of rotation $y'$ and $y''$ bifurcate off of the $y$-axis and rotate in the $xy$-plane towards the $x$-axis. The stability picture is summarized in the following table. We note in particular, that the "saddle" rotation around the $y$-axis splits into two saddles and a sink – as it turns out, this sink is a sink since it (locally) minimizes the energy on the surface $\mathcal{M}_\mu$. Similarly, the $z$-rotation undergoes a pitchfork bifurcation and splits into a saddle and two unstable foci. The detailed analysis of this picture will appear elsewhere. From the more practical point of view one might want to change $\mu$ rather than $k$; it is not hard to see, in fact, that the bifurcation diagram with $\mu$ as a parameter is obtained from the diagram in figure 3.1 by relabeling the $k$-axis into the $\mu$-axis and by reversing the direction of the parameter axis.

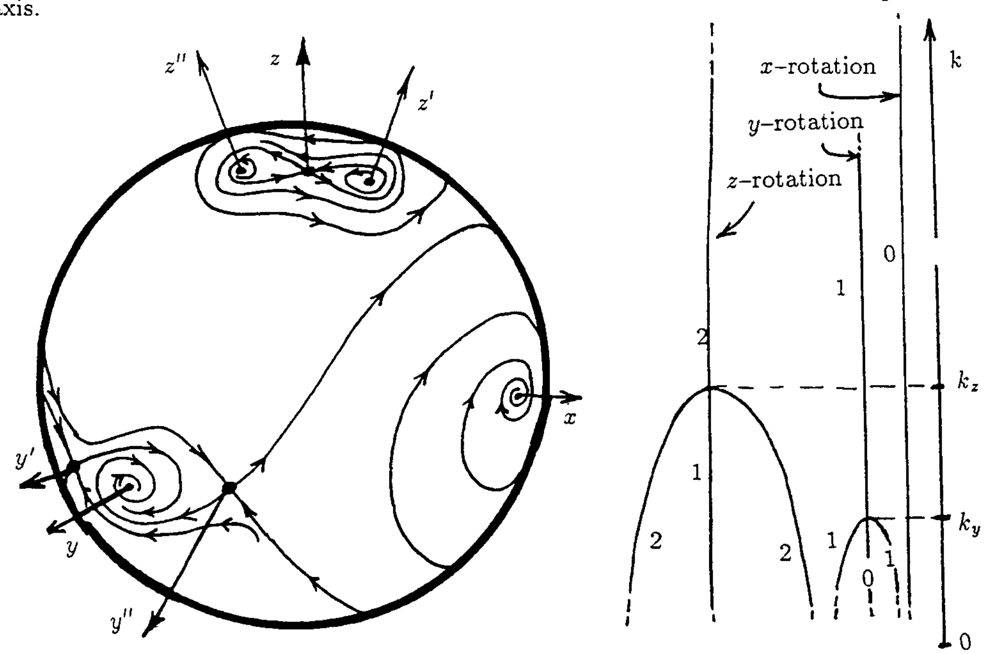


Figure 3.1. Equilibrium rotations of the system in $\mathcal{M}_\mu$ for $k < k_y$ and the bifurcation diagram.

<table>
<tr><th>k / axis</th><th>$0 < k < k_y$</th><th>$k_y < k < k_z$</th><th>$k_z < k < \infty$</th></tr>
<tr><td>$x$</td><td colspan="3">0</td></tr>
<tr><td>$y$</td><td>0</td><td colspan="2">1</td></tr>
<tr><td>$y', y''$</td><td>1</td><td colspan="2">—</td></tr>
<tr><td>$z$</td><td colspan="2">1</td><td>2</td></tr>
<tr><td>$z', z''$</td><td colspan="2">2</td><td>—</td></tr>
</table>

Table 3.2. The entries in the table give the Morse index of the equilibrium points of the flow (2.2) constrained to $\mathcal{M}_\mu$.

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