On a Problem by Arnold on Periodic Motions in Magnetic Fields

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To the memory of Jürgen Moser

1 The Problem and the Result

In this note we show the existence of at least three closed orbits for a charged particle moving on a torus under the influence of a magnetic field. The proof calls for use of Moser's theorem on density transformation; we provide an alternative proof of this theorem, based on the idea of diffusion. For completeness we give a proof of the uniformization theorem for the case of a torus (see the appendix) based on some physical considerations that go back to Riemann, as well as using a recent result of Avellaneda and Lin (also Moser and Struwe).

The result of this note gives a partial solution to a problem posed by Arnold in [1] concerning the existence of at least three closed orbits of a particle whose trajectories on the 2-torus have a prescribed geodesic curvature, the latter computed relative to a given Riemannian metric. We prove this result by reducing it to an equivalent problem of a charged particle moving in the covering plane of the torus in the presence of a magnetic field perpendicular to the torus, under the influence of a potential force:

$$\ddot{z} + iB(x, y)\dot{z} + \nabla V(x, y) = 0, \quad z = x + iy. \tag{1.1}$$

Throughout this note we assume the strength $B(x, y)$ of the magnetic field and the potential $V(x, y)$ to be smooth and periodic with respect to a lattice in the covering plane: $B(z + n_1\mathbf{e}_1 + n_2\mathbf{e}_2) = B(z)$ and $V(z + n_1\mathbf{e}_1 + n_2\mathbf{e}_2) = V(z)$ where $\mathbf{e}_1, \mathbf{e}_2$ is a basis in $\mathbb{R}^2$, and where $z = (x, y)$, $n_i \in \mathbb{Z}$.

We will prove the following:

THEOREM 1.1 *Assume that the energy value E, the magnetic field $B(z)$, and the potential $V(z)$ satisfy*

$$\min_{\mathbb{R}^2} B(z) > \max_{\mathbb{R}^2} \frac{|\nabla V|}{\sqrt{E - V}}. \tag{1.2}$$

Then there exist at least three distinct periodic orbits of (1.1) *with energy E*:

$$\frac{1}{2}|\dot{z}|^2 + V(z) = E. \tag{1.3}$$

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Remark 1.2. Condition (1.2) amounts to the requirement that all trajectories with the energy E are bent in the same direction, i.e., that the Lorentz force $iB\dot{z}$, always normal to the velocity $\dot{z}$, exceed in magnitude the potential force ∇V : $B|\dot{z}| > |\nabla V|$. The last inequality together with the energy relation (1.3) produces (1.2).

In his 1987 paper Ginzburg [6] proved the existence of closed orbits for sufficiently large magnetic fields. Ginzburg's proof is based on the observation that for B large enough (using the notation of (1.1)) a Poincaré map of the flow has a global generating function on the 2-torus.[1] The assumption (1.2) in the present note is somewhat weaker because it does not guarantee the existence of a global generating function. In the absence of such a function we are led to using the result of Conley and Zehnder to prove the existence of periodic orbits. This note allows intermediate values of B, but still requires the lower bound (1.2) on B. Removing this bound altogether remains an open problem. We mention in this connection that Ginzburg has observed in [7] that the existence of (at least one) periodic orbit on the $2n$-torus in an arbitrary magnetic field and with an arbitrary metric for almost all energy values follows from the work by Hofer and Zehnder in [10]. For more recent related results and generalizations we refer to [8, 9].

Remark 1.3. After this note was finished, it came to my attention that an outline of the approach taken here had already been mentioned by V. Kozlov in a personal communication to V. Ginzburg, as mentioned in [6]; see also [11].

1.1 A More General Formulation

We show in this section that the seemingly more special case (1.1) of a particle in the magnetic + potential field in fact includes the general case of an arbitrary Riemannian metric with magnetic field (see [7]):

$$\delta \int (\sqrt{L_2} + L_1)dt = \delta \int \big(\langle A(q)\dot{q}, \dot{q}\rangle^{\frac{1}{2}} + \langle a(q), \dot{q}\rangle\big)dt = 0\,, \tag{1.4}$$

where A is a positive definite matrix and $a(q) = \mathrm{col}(a_1, a_2)$; both A and a are 1-periodic in both components of q, i.e., are defined on the covering plane of the torus $\mathbb{T}_1 = \mathbb{R}^2$ modd 1.

By the uniformization theorem[2] there exists a uniformizing map $q = F(z)$ turning the given Riemannian metric on $\mathbb{T}_1$ into a conformally flat metric on the "skewed" torus $\mathbb{T}_2 = \mathbb{R}^2 \bmod (n_1\mathbf{e}_1 + n_2\mathbf{e}_2)$ with the moduli basis $\mathbf{e}_1$ and $\mathbf{e}_2$: $\langle A\,dq, dq\rangle \equiv \langle dF^{\mathsf{T}} A\, dF\, dz, dz\rangle = \lambda^2(z)\langle dz, dz\rangle$. Thus λ is periodic with respect to the lattice generated by $\mathbf{e}_1$ and $\mathbf{e}_2$. We note that the uniformizing map F^{-1} maps the square lattice $\{(n_1, n_2)\}$ onto a skew lattice $\{n_1\mathbf{e}_1 + n_2\mathbf{e}_2\}$: $F(z + n_1\mathbf{e}_1 + n_2\mathbf{e}_2) = F(z) + (n_1, n_2)$. With an obvious normalization (dilation and scaling) we can

[1] Actually, the results in [6] are more general, dealing with arbitrary even-dimensional compact manifolds without boundary.

[2] See, e.g., ([14]). We give a concise, almost self-contained proof of the theorem in the appendix.

achieve $\mathbf{e}_1 = \mathrm{col}(1, 0)$, so that the lattice depends on two moduli, the coordinates of $\mathbf{e}_2$.

In the new "isothermal" coordinates z the variational problem takes the form

$$\delta \int \left(\lambda\langle \dot{z}, \dot{z}\rangle^{\frac{1}{2}} + \langle b(z), \dot{z}\rangle\right)dt = 0 . \tag{1.5}$$

The Euler-Lagrange equations

$$\frac{d}{dt}\left(\lambda\frac{\dot{z}}{|\dot{z}|}\right) + \left(b_z - b_z^{\mathsf{T}}\right)\dot{z} - |\dot{z}|\nabla\lambda = 0 \tag{1.6}$$

inherit the invariance of the variational problem under arbitrary rescalings of time. We remove this redundancy by selecting solutions with the time parametrization given by $|\dot{z}| = \lambda(z)$. With such a choice, equation (1.6) turns into equation (1.1) where $B = \mathrm{curl}\, b = (b_2)_x - (b_1)_y$ and $V = -\lambda^2/2$. We also note that energy conservation $\dot{z}^2/2 + V(z) = E$ with $E = 0$ amounts to the condition $|\dot{z}| = \lambda$ that we have selected. We thus conclude:

THEOREM 1.4 *Any solution of the variational problem* (1.4) *corresponds to a zero-energy solution of equation* (1.1), *where* $V = \nabla(-\lambda^2/2)$ *and* $\lambda(z)$ *is a conformal factor of the above discussion.*

As a consequence of the above remarks and of the main theorem, we have the following:

COROLLARY 1.5 *The variational problem* (1.5) *on the torus* $\mathbb{T}_2 = \mathbb{R}^2 \bmod n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ *has at least three periodic solutions if*

$$\min \frac{|B(z)|}{|\nabla\lambda(z)|} > \sqrt{2} \quad \textit{where } B = \mathrm{curl}\, b \equiv \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} . \tag{1.7}$$

In conclusion, equation (1.1) of a charged particle in a magnetic field is slightly more general than the variational problem (1.4).

2 Proof of Theorem 1.1

2.1 Plan

(i) Following Arnold [1] we define the mapping of the configuration 2-torus as follows. Restrict our attention to a fixed energy surface (1.3) once and for all.[3] Starting with an arbitrary point z, we shoot the particle in the x-direction and record its position at the first moment when its velocity $\dot{z}$ turns by 2π, calling this new position $\varphi(z)$, Figure 2.1; the map φ is well-defined if (1.2) holds, as we shall see later.

(ii) We will show that the map φ preserves a Liouville measure, $\rho(x, y)d\lambda$, where $d\lambda = dx\, dy$, and that φ fixes the center of mass of the torus with respect to this measure.

[3] This energy surface is a 3-torus; we choose x, y, and $\theta = \arg \dot{z}$ as the coordinates.

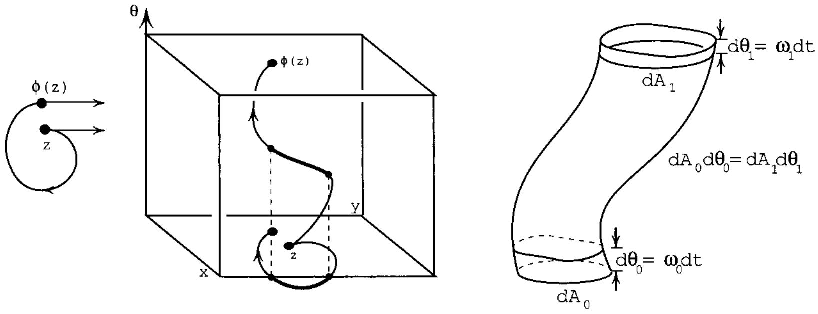

FIGURE 2.1. The invariant measure $\rho(z)$ of the section map.

(iii) Since the measure $\rho\, d\lambda$ is generally not Lebesgue, the Conley-Zehnder fixed point theorem [4] is not directly applicable. With a proper choice of a "homogenizing" diffeomorphism $h : \mathbb{R}^2 \mapsto \mathbb{R}^2$, the conjugate map $\psi = h \circ \varphi \circ h^{-1}$ does become Lebesgue measure-preserving, and we will show furthermore that ψ preserves the center of mass with respect to the Lebesgue measure. The homogenizing diffeomorphism h is provided by a theorem of Moser [12]. We will also give an independent proof of Moser's theorem; this proof is motivated by a physical argument based on heat flow.

2.2 The Angle as New Time

We introduce the polar coordinates r and θ on the velocity space via $\dot{z} = re^{i\theta}$; from the conservation of energy we obtain the speed

$$r = \sqrt{2(E - V(z))} \equiv v(z)$$

as a function of the position. The equations of motion on the energy torus become

$$\begin{cases} \dot{z} = v(z)e^{i\theta} \\ \dot{\theta} = \omega(z, \theta)\,, \end{cases} \tag{2.1}$$

where

$$\omega \equiv -B - \operatorname{Im} \frac{iV'}{v} e^{-i\theta}\,. \tag{2.2}$$

The assumption (1.2) amounts to $|\omega| > 0$; we assume without loss of generality that

$$-\omega(z, \theta) \geq c > 0\,.^4$$

[4] The alternative case reduces to this one via the change of t to $-t$.

As a consequence, the θ-advance maps $\varphi_{\theta_0}^{\theta_1}$ are well-defined for any pair $\theta_0 \leq \theta_1$. Choosing now θ as the new time, we rewrite (2.1) in the form

$$\frac{dz}{d\theta} = \frac{v(z)}{\omega(z,\theta)} e^{i\theta} . \tag{2.3}$$

We observe that the flow of (2.1) preserves the Lebesgue measure $dx \wedge dy \wedge d\theta$, since the divergence of the vector field (2.1) is computed to be zero. The reduced equations (2.3) preserve the measure $\omega(z,\theta)d\lambda$, where $d\lambda = dx\,dy$. Indeed, the Lebesgue volumes swept by the top and the bottom of a flow tube in time dt (see Figure 2.1) by the flow (2.1) are the same since that flow is divergence free; these volumes are $dA_0\,\omega_0\,dt = dA_1\,\omega_1\,dt$ (Q.E.D.), where dA is the Lebesgue area of the base of the tube, implying the desired $\omega_0\,dA_0 = \omega_1\,dA_1$. A more formal restatement of this proof: using the divergence theorem, and letting T be the flow tube with the top and bottom boundaries D_1 and D_0, $D_1 = \varphi D_0$ at $\theta = \theta_0$ and $\theta = \theta_1$, we get

$$0 = \frac{d}{dt}\iiint_T d\lambda\,d\theta = \iint_{D_0} -\omega(z,\theta_0)d\lambda + \iint_{D_1} \omega(z,\theta_1)d\lambda .$$

We conclude that the Poincaré section map $\varphi \equiv \varphi_0^{2\pi}$ preserves the measure $\omega(z,0)d\lambda$ and thus the normalized measure as well:

$$\rho(z)d\lambda = \frac{\omega(z,0)}{[\omega(z,0)]}\,d\lambda . \tag{2.4}$$

More generally, for any slices $\varphi = \alpha$ and $\varphi = \beta$ we have measure preservation between slices: $\iint_D \rho(z,\alpha)d\lambda = \iint_{\varphi_\alpha^\beta D} \rho(z,\beta)d\lambda$, or infinitesimally

$$\rho(z,\alpha) = \rho\big(\varphi_\alpha^\beta z, \beta\big)\det d\varphi_\alpha^\beta(z) . \tag{2.5}$$

Remark 2.1. The mean value $[\omega(z,\theta)] = -[B]$ is θ-independent.

Indeed, the mean value of the ratios V_x/v and V_x/v entering ω (see (2.2)) is zero since these ratios are the x- and y-derivatives, respectively, of the periodic function $2\sqrt{E-V}$.

2.3 Center of Mass Is Fixed by φ in Measure $\rho(z,0)d\lambda$

Recall that $\varphi_\alpha^\theta z$ denotes the solution of (2.3) with $\varphi_\alpha^\alpha = \mathrm{id}$. Consider the position of the center of mass carried with the flow:

$$Z(\theta) = \iint_{\varphi_0^\theta Q} z\rho(z,\theta)d\lambda ,$$

where Q is the fundamental domain of the lattice, $Q = \{ue_1 + ve_2 : 0 \leq u < 1, 0 \leq v < 1\}$. Our goal is to show that

$$Z(2\pi) = Z(0) . \tag{2.6}$$

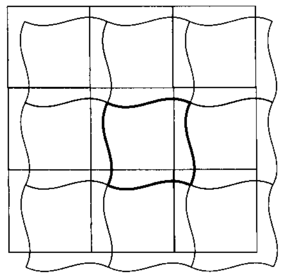

FIGURE 2.2. $Q = \cup T_{\mathbf{n}\in\mathbb{Z}^2}\psi(Q)\cap Q$.

Wishing to find $Z'(\theta)$, we substitute $z = \varphi_0^\theta(w)$, obtaining

$$Z(\theta) = \iint\limits_{Q} \varphi_0^\theta w\big(\rho(\varphi_0^\theta w, \theta)\det d\varphi_0^\theta(w)\big)d\lambda\,.$$

The expression in parentheses is θ-independent, as follows from (2.5). Differentiating by θ, using (2.3), and returning to $z = \varphi_0^\theta(w)$ results in

$$\begin{aligned} Z'(\theta) &= e^{i\theta}\iint\limits_{\varphi_0^\theta Q}\frac{v(z)}{\omega(z,\theta)}\rho(z,\theta)d\lambda \\ &= e^{i\theta}[B]^{-1}\iint\limits_{\varphi_0^\theta Q} v(z)d\lambda = e^{i\theta}[B]^{-1}\iint\limits_{Q} v(z)d\lambda\,, \end{aligned} \tag{2.7}$$

where the last equality follows from (i) the fact that $\phi_0^\theta Q$ is congruent to Q modulo integer translations and (ii) the invariance of $v(z)$ under these translations. Integration by θ gives (2.6).

2.4 Homogenizing the Invariant Measure

To apply the Conley-Zehnder fixed point theorem [4, corollary 2] we introduce new variables in which the invariant measure $\rho(z)d\lambda$ turns into the Lebesgue measure λ. We thus are looking for a diffeomorphism h of the covering plane such that $\iint_D \rho(z)d\lambda = \iint_{h(D)} d\lambda$ for any domain D or, equivalently, for a solution h of the nonlinear PDE

$$\det h'(z) = \rho(z)\,. \tag{2.8}$$

With such h the conjugate map $\psi = h\circ\varphi\circ h^{-1}$ preserves the Lebesgue measure.

The desired map h exists by a theorem by Moser [12]. We give here a different proof of this theorem, which may be of independent interest, for the particular case of a torus. As a side remark, we note that Moser's result is more general because it has been carried out in the case of an arbitrary Riemannian manifold without boundary; for the case of manifolds with boundary we refer to Dacorogna and Moser [5], where the existence of a map fixing the boundary and having a prescribed Jacobian determinant is proven. We mention also a remarkable combinatorial proof due to Burago and Kleiner [3] that gives a deeper insight into the problem of regularity (proving, e.g., that for some continuous ρ there is no bi-Lipschitz map h satisfying equation (2.8)).

Construction of *h* Satisfying (2.8)

Our construction comes from a simple idea: Consider a homogeneous porous material permeated by gas of variable density $\rho(z,t)$. The gas diffuses from denser to sparser areas in some way, to be specified later. The diffusion will equalize the density from its initial distribution $\rho(z,0) = \rho(z)$ to its final constant density state $\lim_{t\to\infty} \rho(z,t) = [\rho(z)] = \text{const} = 1$ (by fixing total mass). The desired map h is simply the assignment of a particle's initial position to its final position. The conservation of mass will then assure that the original and final areas dA_0 and dA_1 of a "blob" have the same mass: $\rho dA_0 = 1 \cdot dA_1$, i.e., $\rho = dA_1/dA_0 = \det h'$, as desired. We make this heuristic idea more precise.

We define $\rho(z,t)$ as the solution of the heat equation

$$\rho_t = \Delta\rho \quad \text{with the initial condition } \rho(0,z) = \rho(z)\,, \tag{2.9}$$

and with $\rho(z,t)$ thus specified, define the motion of "particles" via the (nonautonomous) ODE

$$\dot{z} = -\frac{1}{\rho}\nabla\rho \equiv -\nabla \ln \rho\,, \tag{2.10}$$

and finally define $h := z_{t=0} \mapsto z_{t=\infty}$ for solutions of the last ODE.[5] This completes the construction of h; it remains to prove that h is well-defined and that it solves the PDE (2.8).

THEOREM 2.2 *If $\rho \in C^3(\mathbb{T}_2)$, then the map h defined above is a diffeomorphism and it solves the partial differential equation* (2.8).

Remark 2.3. (See [5]) Solutions to (2.8) are nonunique and can be changed by an area-preserving diffeomorphism g, $\det g = 1$: If h satisfies (2.8), then so does $g \circ h$. This freedom in the choice of a particular h manifests in much unused freedom in our diffusion construction. For instance, the same idea works with anisotropic diffusion given by a positive definite matrix A (allowed to depend on z) and with an arbitrary equation of state $p = f(\rho)$ (where p is pressure) with a monotone f. The evolution of the density and the motion of the particles would then be given by $\rho_t = \operatorname{div}(A\nabla p)$, $p = f(\rho)$, and $\dot{z} = -\frac{1}{\rho}A\nabla p$, respectively. The flow thus defined still preserves the measure ρ.

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[5] The factor $1/\rho$ is required to respect the conservation of mass (which is equivalent to (2.8)): $\operatorname{div}(\rho\mathbf{v}) = -\operatorname{div}(\nabla\rho) = -\Delta\rho = -\rho_t$, so that the mass conservation $\rho_t + \operatorname{div}(\rho\mathbf{v}) = 0$ is satisfied. In other words, the t-advance map of (2.10) carries the measure $\rho(z)d\lambda$ into the measure $\rho(z,t)d\lambda$.

PROOF OF THEOREM 2.2:

(i) h satisfies (2.8).

The key point is the observation that (below $\rho_z \equiv \nabla\rho$)

$$\begin{aligned}\frac{d}{dt}\ln\rho(h^t z, t) &= \frac{\rho_t}{\rho} + \frac{\rho_z \cdot \dot{z}}{\rho} = \frac{\Delta\rho}{\rho} - \frac{\rho_z^2}{\rho^2} \\ &= \left(\frac{\rho_x}{\rho}\right)_x + \left(\frac{\rho_y}{\rho}\right)_y = \operatorname{tr}\left(\frac{\rho_z}{\rho}\right)_z,\end{aligned} \tag{2.11}$$

where $\rho = \rho(h^t z, t)$ throughout. On the other hand, the Jacobian $\frac{\partial}{\partial z}h^t z = H(t, z)$ satisfies the linearization of (2.10):

$$\frac{\partial}{\partial t}H = -\left(\frac{\rho_z}{\rho}\right)_z H, \tag{2.12}$$

and thus by a theorem of Abel

$$\frac{d}{dt}\ln\det H = -\operatorname{tr}\left(\frac{\rho_z}{\rho}\right)_z. \tag{2.13}$$

Comparing this with (2.11), we conclude that $\frac{d}{dt}(\rho(h^t z, t)\det H) = 0$. Since $\det H(0, z) = 1$, we have

$$\rho(z, 0) = \rho(h(z), \infty)\det h'(z), \quad h(z) \equiv h^\infty(z).$$

This becomes (2.8) once we recall that $\rho(z, 0) = \rho(z)$ and observe that $\rho(z, \infty) = [\rho] = 1$.

(ii) $h(z)$ is a diffeomorphism.

First, $h^\infty z \equiv \lim_{t\to\infty}\phi^t z$ is well-defined since the integral in

$$h^\infty z = z + \int_0^\infty \frac{\rho_z(h^s z, s)}{\rho(h^s z, s)}\, ds$$

converges because the integrand decays exponentially in s since all the derivatives of the solutions to (2.9) do and since

$$\rho(z, t) \geq \min_Q \rho > 0 \quad \forall z \text{ and } \forall t \geq 0.$$

Differentiability (and invertibility) of $h^\infty z$ follows from considering the Jacobian $H(t, z) = \partial_z h^t z$, which satisfies (2.12). Since the coefficient matrix of (2.12) decays exponentially, we conclude that $H(\infty) = \lim_{t\to\infty} H(t) < \infty$; moreover, $\det H(\infty) = \exp\int_0^\infty \operatorname{tr}(\rho_z/\rho)ds > 0$ since the integral is finite. We showed that h is a uniform limit in C^1 of diffeomorphisms h^t, and thus is itself a diffeomorphism. The proof of Theorem 2.2 is complete. □

Having thus constructed h, we show the conjugate map ψ satisfies the conditions of the Conley-Zehnder theorem. The property $\det\psi = 1$ is obvious (from (2.8) and (2.5)), and it remains to show that ψ fixes the center of mass of Q.

2.5 Map $\psi = h \circ \varphi \circ h^{-1}$ Fixes Center of Mass of Q

Notation. For a measurable set S, we denote

$$[S] = \iint_S z\,d\lambda\,, \quad [S]_\rho = \iint_S z\rho(z)\,d\lambda\,, \quad \text{and} \quad \delta(S) = [S] - [h^{-1}S]_\rho\,.$$

We prove that the Lebesgue center of mass is preserved under ψ:

$$[Q] = [\psi Q]\,. \tag{2.14}$$

To that end, consider the relative position of the centers of mass with respect to $d\lambda$ and $\rho\,d\lambda$:

$$\delta(Q) = [Q] - [h^{-1}Q]_\rho\,. \tag{2.15}$$

LEMMA 2.4

$$\delta(Q) = \delta(\psi(Q))\,. \tag{2.16}$$

PROOF OF (2.16): The proof relies on the property $h \circ T^{\mathbf{n}} = T^{\mathbf{n}} \circ h$, where $T^{\mathbf{n}}$ is the integer translation by $\mathbf{n} = n_1 e_1 + n_2 e_2$ in the z-plane. Let $Q_{\mathbf{n}} = T^{\mathbf{n}} Q$. We have

$$\psi Q = \bigcup_{\mathbf{n}\in\mathbb{Z}^2} \psi(Q) \cap Q_{\mathbf{n}}\,. \tag{2.17}$$

Since the union is disjoint, we obtain by using $\delta(T^{\mathbf{n}}S) = \delta(S)$ for any measurable set S in the second step,

$$\delta(\psi Q) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \delta(\psi Q \cap Q_{\mathbf{n}}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \delta(T^{-\mathbf{n}}\psi Q) \cap Q) = \delta(Q)\,,$$

where the last step uses the fact that Q is the disjoint union of the sets $T^{\mathbf{n}}\psi(Q)) \cap Q$. The proof of (2.16) is complete. □

Finally, to prove (2.14), we rewrite (2.16) as

$$[Q] - [\psi Q] = [h^{-1}Q]_\rho - [h^{-1}\psi Q]_\rho \tag{2.18}$$

and show that the right-hand side vanishes. Via $h^{-1} \circ \psi = \varphi \circ h^{-1}$, the right-hand side becomes

$$[h^{-1}Q]_\rho - [\varphi h^{-1}Q]_\rho\,. \tag{2.19}$$

Observe that the sets $h^{-1}Q$ and Q are congruent modulo $T^{\mathbf{n}}$ since h is the lift of a toral isomorphism. According to the lemma below, (2.19) does not change upon replacing $h^{-1}Q$ by a congruent set Q; (2.19) becomes

$$[Q]_\rho - [\psi Q]_\rho = 0\,,$$

the last equality following from the preservation of the center of mass (2.6). The proof of (2.14) is complete modulo the following lemma.

LEMMA 2.5 *If a measurable set $S \subset \mathbb{R}^2$ is congruent to the fundamental parallelogram Q modulo $T^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2$, then for any lift φ of a toral diffeomorphism we have*

$$[S]_\rho - [\varphi S]_\rho = [Q]_\rho - [\varphi Q]_\rho \,. \tag{2.20}$$

PROOF: With $\mathbf{n} \in \mathbb{Z}^2$ define $S_{\mathbf{n}} = S \cap Q_{\mathbf{n}}$, and observe that $S = \bigcup S_{\mathbf{n}}$ (using that Q is a fundamental domain), and that the union is disjoint. We have

$$[S]_\rho = \sum_{\mathbf{n}\in\mathbb{Z}^2} [S_{\mathbf{n}}]_\rho = \Big[\bigcup T^{-\mathbf{n}} S_{\mathbf{n}}\Big]_\rho + \sum \mathbf{n} \iint_{T^{-\mathbf{n}} S_{\mathbf{n}}} \rho \, d\lambda \,. \tag{2.21}$$

Similarly, we have

$$[\varphi S]_\rho = \sum_{\mathbf{n}\in\mathbb{Z}^2} [\varphi S_{\mathbf{n}}]_\rho = \Big[\bigcup T^{-\mathbf{n}} \varphi S_{\mathbf{n}}\Big]_\rho + \sum \mathbf{n} \iint_{T^{-\mathbf{n}} \varphi S_{\mathbf{n}}} \rho \, d\lambda . \tag{2.22}$$

We substitute $\bigcup T^{-\mathbf{n}} S_{\mathbf{n}} = Q$ and $\bigcup T^{-\mathbf{n}} \varphi(S_{\mathbf{n}}) = \varphi(\bigcup T^{-\mathbf{n}} S_{\mathbf{n}}) = \varphi(Q)$ for the domains of integration in the integrals in (2.21) and (2.22). Observe also that the last terms in (2.21) and in (2.22) coincide since $T \circ \varphi = \varphi \circ T$ and since φ preserves the measure $\rho \, d\lambda$. Subtracting (2.22) from (2.21), we obtain (2.20), and the proof of Lemma 2.5 is complete. □

Appendix: Uniformization

A.1 Formulation.

In this subsection we give a short proof of the existence of a transformation in which the metric becomes conformally flat, i.e., conformal to the Euclidean metric. The proof uses a recent result of Avellaneda and Lin.

Below $A(z)$ stands for a positive definite 2×2 matrix smooth in z on the covering plane of a torus.

THEOREM A.1 *For any Riemannian metric $ds^2 = \langle A(z)dz, dz \rangle$ on the torus $\mathbb{T}_1 = \mathbb{R}^2 \operatorname{mod} (m, n)$, there exist*

(i) *two real numbers (the moduli) a and b defining the torus $\mathbb{T}_2 = \mathbb{R}^2 \operatorname{mod} me_1 + ne_2$ with $e_1 = (1, 0)$ and $e_2 = (a, b)$, $b \neq 0$,*

(ii) *a positive scalar function $\lambda(z) > 0$, and*

(iii) *the map $w = F(z)$ such that $F : \mathbb{T}_1 \mapsto \mathbb{T}_2$ and*

$$ds^2 = \langle A(z)dz, dz \rangle = \lambda(z) \langle dw, dw \rangle \,, \quad dw = F' dz \,, \tag{A.1}$$

or equivalently, by manipulating $A = \lambda (F')^{\mathsf{T}} F'$:

$$F' A^{-1} (F')^{\mathsf{T}} = \lambda^{-1} I \,, \tag{A.2}$$

or, in the notation $F = \operatorname{col}(u, v)$ and $\langle w, w' \rangle_{A^{-1}} = \langle A^{-1} w, w' \rangle$,

$$\langle \nabla u, \nabla u \rangle_{A^{-1}} = \langle \nabla v, \nabla v \rangle_{A^{-1}} = \lambda^{-1} \,, \quad \langle \nabla u, \nabla v \rangle_{A^{-1}} = 0 \,. \tag{A.3}$$

A.2 Proof of the Uniformization Theorem

Local Construction of the Map F

The desired transformation $F = \operatorname{col}(u, v) \equiv (u, v)^{\mathrm{T}}$ is constructed from a physical argument as follows. Let us treat the plane as a heat-conducting medium in which a steady temperature $u = u(z)$ is maintained. The medium is anisotropic: The heat flux $\mathbf{f} = -S\nabla u$, where $S > 0$, is a symmetric (conductivity) matrix, to be chosen later according to the Riemannian metric. We define the conjugate function v as the heat flux through a curve ending at z,

$$v(z) = -\int_0^z (S\nabla u) \cdot \mathbf{N}\, ds = \int_0^z (JS\nabla u) \cdot T\, ds \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

differentiation gives

$$\nabla v = JS\nabla u\,. \tag{A.4}$$

For v to be well-defined, the integral must be path-independent:

$$\operatorname{div} S\nabla u = 0\,; \tag{A.5}$$

i.e., u has to solve the Beltrami equation.

Our choice of S is dictated by the need to satisfy equations (A.3). These equations are equivalent to one matrix equation

$$\sqrt{A^{-1}}\,\nabla v = J\big(\sqrt{A^{-1}}\,\nabla u\big) \tag{A.6}$$

since both express the orthogonality and equality of (Euclidean) lengths of the vectors $\sqrt{A^{-1}}\,\nabla u$ and $\sqrt{A^{-1}}\,\nabla v$. Comparison of equation (A.6) with equation (A.4) forces $JS = \sqrt{A}\,J\sqrt{A^{-1}}$, or $S = (J^{-1}\sqrt{A}\,J)\sqrt{A^{-1}} = \sqrt{\det A}\,A^{-1}$, where the last step uses the lemma in the next paragraph. We thus showed (modulo the next lemma) that with such a choice of S and with u and v defined as above, the map F uniformizes the metric, i.e., satisfies (A.3). This construction provides a local existence of F; one just needs to choose a solution u of the Beltrami equation with $\nabla u \neq 0$ in a neighborhood of a point.

LEMMA A.2 *Let $M > 0$ be a 2×2 real (symmetric) matrix. Then*

$$J^{-1}MJ = M^{-1}\det M\,. \tag{A.7}$$

PROOF: If $\det M = 1$, then (A.7) is evident from the fact that M is a hyperbolic rotation with mutually orthogonal eigendirections and with reciprocal eigenvalues which the conjugation by J permutes, resulting in the inverse matrix. The general case of $\det M \neq 1$ reduces to the last one by rescaling $N = M/\sqrt{\det M}$, so that $\det N = 1$. □

Global Part

We have to show that there exists a solution u of equation (A.5) defined on the whole of $\mathbb{R}^2$ such that $F = \operatorname{col}(u, v)$ is a global diffeomorphism of the covering plane of $\mathbb{T}_1$ onto the covering plane of another torus $\mathbb{T}_2$. According to [13, 2], given any linear function $ax + by$ there exists a periodic function $p(x, y)$ such that $u = ax + by + p(x, y)$ is a solution of (A.5). In particular, there exists a solution of (A.5) of the form $u = x + p(x, y)$. We note that for this solution $\nabla u \neq 0$ for all z, as we will show in the next paragraph. From this we conclude that the level lines of u foliate the entire plane, and through each point there passes exactly one level curve of u. From (A.4) the same holds for v. This shows that F is one-to-one. From (A.4) it also follows that $\det F' \neq 0$.

To show that $\nabla u \neq 0$ for all z, we consider the set of critical points of u in the fundamental square. The points are isolated, and moreover the Poincaré index of each of these points is ≤ -1 since u is a solution of an elliptic equation. On the other hand, by the periodicity of ∇u in x and y, the total index of the vector field ∇u over the fundamental square is zero. This proves that the set of critical points of u is empty.

It remains to show that F is a lift of a torus map, i.e., that there exists a basis $\mathbf{e}_1, \mathbf{e}_2$ of $\mathbb{R}^2$ such that $F(x+1, y) - F(x, y) = \mathbf{e}_1$, $F(x, y+1) - F(x, y) = \mathbf{e}_2$. Indeed, we have $u(x+1, y) - u(x, y) = 1$ and $u(x, y+1) - u(x, y) = 0$ from the construction of u. Furthermore,

$$\begin{aligned} v(x+1, y) - v(x, y) &= \int_{x,y}^{x+1,y} S\nabla u \cdot \mathbf{N}\, ds \\ &= \int_{(0,0)}^{(1,0)} S\nabla u \cdot \mathbf{N}\, ds = v(1,0) - v(0,0) \end{aligned}$$

is independent of (x, y); the same holds for $v(x, y+1) - v(x, y) = v(0, 1) - v(0, 0)$. Thus $\mathbf{e}_1 = (1, a)$ and $\mathbf{e}_2 = (0, b)$ for some $b \neq 0$. This completes the proof of the uniformization theorem.

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