Riemann Mapping by Steepest Descent

We place a heat sink at $z=0$, drawing 2π calories per second, and maintain $u=0$ on ∂D. Figure 2 shows the graph of the resulting stationary temperature distribution.

This is to replace the crossed-out paragraph on the left.

MATHEMATICAL CURIOSITIES

By Mark Levi

In this issue's column we outline a quick constructive proof of the Riemann mapping theorem.

Here's a statement of the theorem: Any simply connected open set D in the plane that is not the entire plane can be mapped conformally and one-to-one onto an open unit disk, taking any point $p \in D$ to the disk's center, with the derivative at p being real and positive.

Our map is produced via a physically motivated argument: We think of a heat-conducting plate D, insulated everywhere except for the boundary, as shown in Figure 1. ~~Figure 2 shows the graph of a stationary temperature distribution u on D, with $u = 0$ on ∂D, and with the heat flux [[into APPROACHING?]] 0 equal to 2π calories per second.~~ Formally, we define

$$u(z) = \ln|z| + u_0(z), \qquad (1)$$

where $u_0(z)$ is a *harmonic function* with boundary conditions chosen to cancel the logarithm on ∂D.*

Level curves of (1) are approximately circles near $z = 0$; more precisely, the set $D_t = \{u \le -t\}$ for large t is approximately a small disk $|z| \le e^{-t-u_0(0)}$ (see Figure 2).

Remarkably, the flow φ^t of the modified gradient field

$$\dot{z} = -\frac{1}{|\nabla u|^2}\nabla u \qquad (2)$$

shrinks D into D_t and does so conformally—i.e., φ^t is almost the desired map!

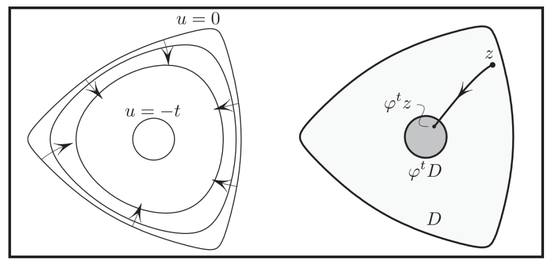

Figure 2. *Constructing the conformal map from D to a disk. Left, the modified gradient flow preserves level curves; right, mapping by the modified gradient flow.*

Indeed, $du/dt = -1$ along (2), and thus $\varphi^t D = D_t$. And because the right-hand side of (2) is analytic,† φ^t is conformal. By dilating D_t we obtain the desired map f in the limit:

$$f(z) = \lim_{t\to\infty} e^t \phi^t z. \qquad (3)$$

The missing details of the proof, which are routine, can be found in ~~the *American Mathematical Monthly*~~ Amer. Math. Monthly 114(2007), no. 3, pp. 246-251.

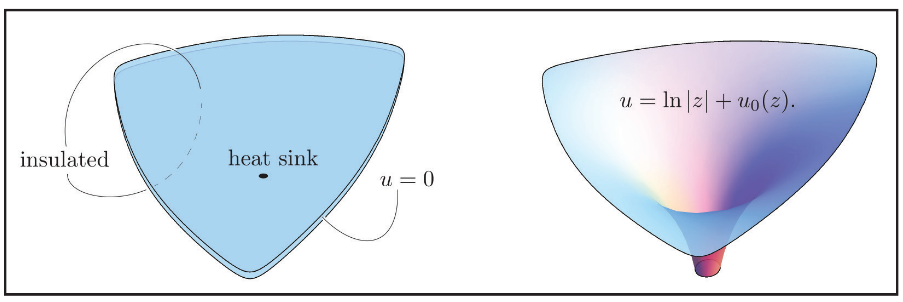

Figure 1.

*Here we use the existence of solutions of the Dirichlet problem, which limits some generality on D. Details can be found in [].NEED THE REFERENCE F. John, Partial Differential Equations, Fourth Edition, Springer.

since it

† ~~It~~ can be written as $1\sqrt{\nabla u} = 1/(u_x - iu_y)$, where $U = u_x$ and $V = -u_y$ satisfy the Cauchy–Riemann equations because u is harmonic.

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