A Simple Derivation of Heron's Formula

MATHEMATICAL CURIOSITIES
By Mark Levi

Heron's formula gives the area A of a triangle with sides a, b, c:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \qquad (1)$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter. Most proofs hide the simple reason for this result. This reason is twofold:

1. An observation that A^2 is a polynomial of degree 4 in a, b, c.
2. $A^2 = 0$ if the triangle degenerates into a point or a segment, i.e., if $a+b+c=0$ or if $a+b-c$ or any of its cyclic permutations vanish.

Taking (1) for granted for the moment, (2) implies that

$$A^2 = k(a+b+c)(a+b-c)(b+c-a)(c+a-b), \qquad (2)$$

where the unknown constant k is independent of a, b, c. To find k, we apply (2) to the right triangle with sides 1, 1, $\sqrt{2}$, thus obtaining

$$\left(\frac{1}{2}\right)^2 = k(2+\sqrt{2})(2-\sqrt{2})\sqrt{2}\sqrt{2}$$

or $k = 1/2^4$. With this value, (2) becomes Heron's formula (1).

To justify (1), we write

$$A^2 = a^2b^2\sin^2\theta = a^2b^2 - a^2b^2\cos^2\theta,$$

where by the theorem of cosines $4a^2b^2\cos^2\theta = (c^2-a^2-b^2)^2$. Observation (2) allowed us to avoid the algebra of factoring A^2.

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