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A "bicycle wheel" proof of the Gauss-Bonnet theorem

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Abstract. 1. A new proof of the Gauss-Bonnet theorem based on its reduction to a theorem about dual cones is given. This reduction was motivated by a simple observation on the motion of rigid bodies. 2. Two different proofs of the dual cones theorem are given, one based on "moving fronts" and another on a heuristic mechanical analogy. 3. As a corollary of this approach we obtain the observation that the total geodesic curvature of a curve on a surface is an invariant of the Gauss map. 4. The "bicycle wheel" idea is applied (a) to construct a "spherimeter", i.e. a mechanical device which calculates areas of spherical regions; (b) to describe an aspect of wave propagation in a waveguide, such as an optical fiber, and (c) to describe twisting of an elastic object such as a beam or a rope. 5. Finally, the relationship with the writhing number and the Berry's phase is described.

Contents

1. Dual Cones and Bicycle Wheels

This note started with the question which arose after changing a flat bicycle tire: can one turn a bicycle wheel (with frictionless bearings) around its axis holding it by only the axis? A deeper look at this problem was stimulated by the realization that the question comes up in some other contexts such as the propagation of light through an optical fiber [C], the motion of rigid bodies [M], in the geometry of twisted beams, springs, ropes and other linear elastic objects, and in some other classical and quantum mechanical problems

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where the Berry-Hannay angles arise [BH], [B], [GKM], [H], [KS], [L], [MMR], [M], [TS].

1.1 A motivating problem

Consider a rigid body such as a balanced bicycle wheel held by the axis. There is no friction in the bearings and both the wheel and the axis are initially at rest. The center of the wheel is assumed to be at a fixed point in space at all times. The axis is made to describe a closed cone (generally noncircular), so that the wheel starts and ends in the same plane. Despite the fact that the component of applied torque in the axial direction is zero, the wheel may find itself turned through an angle $\alpha \neq 0$ with respect to its initial position, see figure 1.1, where $\alpha = \pi/2$.

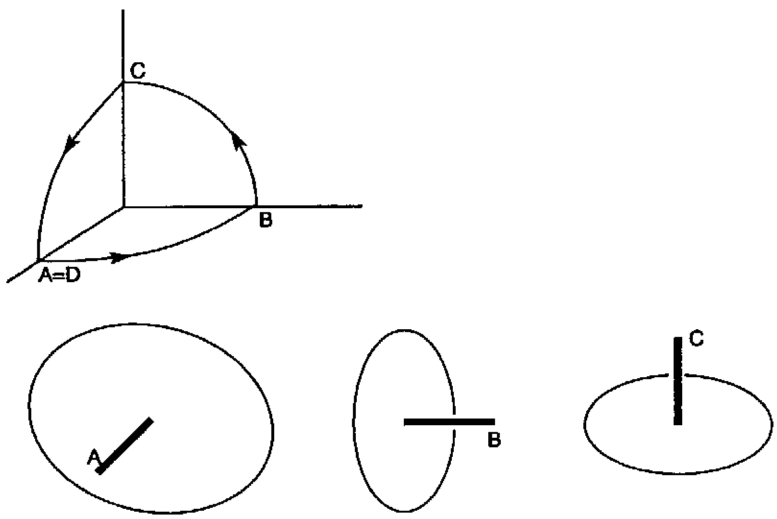


Figure 1.1. A way to turn the wheel by $\frac{\pi}{2}$ without applying torques in the axial direction. A point P on the axis describes a spherical triangle ABCA.

The angle α is perhaps the simplest example of the Hannay-Berry angle [BH], [B], [S]. The next two sections give two different ways to compute α for the above example. As a byproduct we will obtain a new proof of the Gauss-Bonnet theorem, together with several other applications.

A remark on parallel transport. The wheel provides a physical realization of parallel transport of a vector tangent to a surface along a curve on that surface in the following way. Let us keep the plane of the wheel tangent to a given surface with the wheel's center touching the surface at all times, let us set the angular velocity of the wheel in the direction of its axis to be zero (at the outset and thus forever, since the bearings are frictionless), and let us guide the center of the wheel along a curve drawn on the surface. Under these assumptions each spoke of the wheel (viewed as a tangent vector to the surface) undergoes parallel transport. Figure 1.1 illustrates the result of the parallel transport along the path ABCA (although in the figure the wheel is kept at the origin).

1.2 Dual cones and Hannay-Berry angles

Dual cones. Given a cone C in $\mathbb{R}^3$ (Figure 1.2) we define the dual cone C^* as the *envelope of the family of planes passing through the origin* and normal to the generators of C.

This “definition” is, however, ambiguous because it specifies neither the direction of the cone nor one of the two possible solid angles bounded by the cone interior. In the future we will refer to the cone and to the curve of its intersection with the unit sphere interchangeably. Giving the curve a preferred orientation specifies the interior disk as the disk to the left of the curve.

Definition 1. *Let c be a closed oriented curve on the unit sphere traced out by a unit vector $n(t)$, $0 \le t \le 1$, with the orientation given by a unit tangent vector $T(t)$ (so that $T = \frac{\dot n}{|\dot n|}$ or $T = -\frac{\dot n}{|\dot n|}$. In short, $c = (n,\, T)$). We define the dual curve by*

$$c^* \equiv (n^*, T^*) = (T \times n, -T),$$

where $\times$ is the usual cross product in $\mathbb{R}^n$. In other words, the dual curve has the orientation $-T$ and it consists of the points $T \times n$.

Remarks.

1. To justify this definition we make sure that $-T$ is tangent to the curve $n^*(t) = T \times n$: indeed, $(d/dt)(T \times n) = \dot T \times n + T \times \dot n = \dot T \times n \uparrow\uparrow T$; in the last step we used the facts that $\dot T \perp T$ (since $|T| = 1$) and $n \perp T$.

2. Duality is reflexive: $(c^*)^* = c$. Indeed,

$$(n, T) \xrightarrow{*} (T \times n, -T) \xrightarrow{*} ((-\dot T) \times (T \times n, T)) = (n, T)\,.$$

The reflexive property would not hold without the reversal of orientation of the dual curve.

The next definition defines the precise meaning of the oriented area enclosed by the curve and the oriented length of the curve. The reader may wish to skip the details of this definition in the first reading.

Definition 2: the length $L(c)$ and the area $A(c)$. The length $L(c) \equiv L(C)$ of a curve c (or a cone C) is given by

$$L(c) \equiv L(n,T) = \int_0^1 (n \times \dot{n}) \cdot (n \times T)\, dt. \tag{1.1}$$

Parametrizing the disk inside the curve c by $n(t,\tau)$, with $0 \leq t < 1$, $0 \leq \tau \leq 1$ and with $n(t) = n(t,0)$ corresponding to the boundary, we define the area of the disk, i.e. the solid angle of the cone, by

$$A(c) = \int_0^1 \int_0^1 n_\tau \times n_t \cdot n \, d\tau\, dt. \tag{1.2}$$

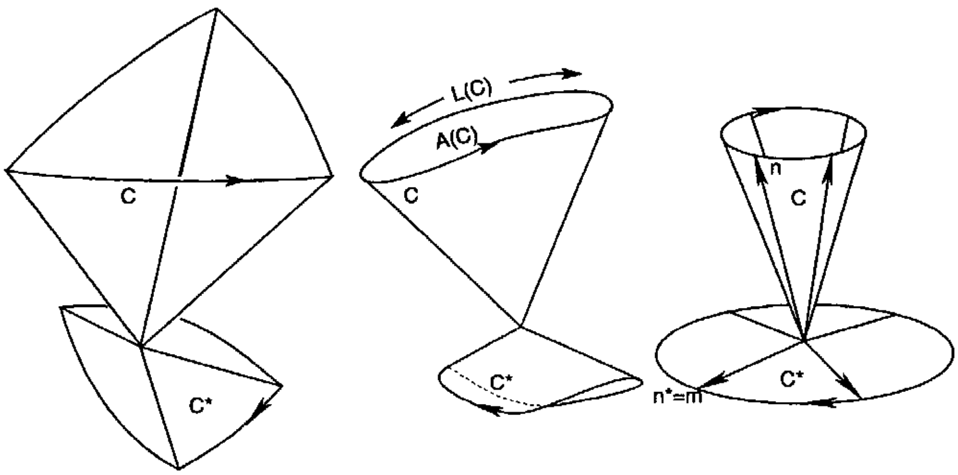


Figure 1.2. Some examples of dual cones.

Coming back to the original question, we ask: by how much does the wheel turn, i.e. how to calculate the Hannay-Berry angle α? The answer is given by

Lemma 1.1. *When the semiaxis of the wheel traces out the closed curve c on the unit sphere, the wheel turns*

$$\alpha = 2\pi - L(c^*). \tag{1.3}$$

The positive direction for α is given by the direction of the semiaxis of the wheel.

The lemma is illustrated in Figure 1.1 with $\alpha = \frac{\pi}{2}$ and $L(c^*) = \frac{3\pi}{2}$. is clear by inspection; this is in agreement with (1.3).

Proof of lemma 1.1 is based on the key observation, figure 1.3: as the axis of the wheel traces out the cone C,

$$\text{the wheel's plane rolls without sliding along the dual cone } C^*, \tag{1.4}$$

that is, 1) the the wheel's plane is tangent to the cone C^* and that 2) the velocity of the points of the wheel which are in contact with the cone C^* is zero. We verify 1) at once by noting that the tangent plane to C^* is given by the vectors $n^* = T \times n \perp n$ and $-T \perp n$, n being the wheel's axis. To prove 2) we observe that $n^* = T \times n$ is collinear with the angular velocity $\Omega = n \times \dot{n}$ of the wheel; the points of a rigid body lying on the line of angular velocity have zero speed.

To verify the angle-length relationship (1.3), we consider one cycle in which the wheel rolls exactly once around the cone C^* and concentrate our attention on the set of points of the wheel which came in contact with the cone. Since there is no sliding, this set is a sector of angle $L(C^*)$. The wheel thus turns through the angle of magnitude $L(C^*)$; taking into account our orientation convention from the statement of the lemma, the angle is $-L(C^*)$, figure 1.3. The angle is actually defined mod 2π, and we choose $\alpha = 2\pi - L(C^*)$ since this gives $\alpha = 0$ in the case when the cone C is a ray corresponding to the case when the wheel has not been moved at all. ◇

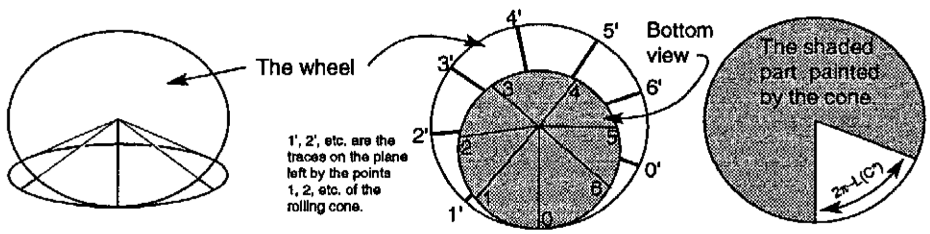


Figure 1.3. The plane of the wheel rolls without sliding along the cone C^* dual to the semiaxis cone C.

Now the question arises: can one compute α directly from the cone C, rather than from its dual? In other words, what is the relationship between the dual cones? The answer to this question is given by a geometrical theorem of the next section. The theorem says in effect that the solid angle of a cone plus the length (of the intersection with the unit sphere) of its dual equals 2π. This simple-sounding fact is equivalent to the Gauss-Bonnet theorem (as will be explained) and provides a simple new proof to it.

2. A theorem on dual cones

Theorem 2.1 (the dual cones' theorem). *The sum of the solid angle of a cone and of the length of its dual equals* 2π*:*

$$A(c) + L(c^*) = 2\pi\,; \tag{2.1}$$

the precise definitions of the solid angle and the length are given below.

Remark 2.1. It may seem strange that one adds the lengths and the areas in eq. (2.1), but this is only because we work with the unit sphere; for a sphere of radius R eq. (2.1) changes into $A(c) + RL(c^*) = 2\pi R^2$, which has the right dimension count.

Corollary 2.1. *The Hannay-Berry angle* α *of the wheel is given by the solid angle* $\alpha = A(C)$*, of the cone* C *described by a semiaxis of the wheel. Indeed,* $\alpha = 2\pi - L(c^*) = A(c)$*.* ◇

Remark 2.2. Eq. (2.1) is actually equivalent to the Gauss-Bonnet theorem on the sphere: $A(c)$ is the integral of the Gaussian curvature and $L(c^*)$ turns out to be precisely the integral of the geodesic curvature (this is explained in more detail later.) It should be noted that the dual cones theorem does not mention curvatures.

In the next section we give a heuristic proof of the dual cones theorem, followed by a rigorous proof based on the deformation of curves in section 2.2.

2.1 An energy "proof" by mechanical analogy.

Our aim now is to provide an intuitive insight into (2.1); in this section we sacrifice rigor when it gets in our way.

We think of the dual curves c, c^* as sets of endpoints of a "bouquet of frames" $(n(t), n^*(t))$, $0 \le t \le 1$, figure 2.1, each frame being a pair of rods of unit length joined rigidly to one another at the right angle, with the joint of each pair (n, n^*) held at the origin in $\mathbb{R}^3$. Let us imagine that 1) the spherical domain enclosed by the curve c is filled with a *two dimensional* gas with negative pressure $p = -1$, and 2) the dual curve c^* is made out of a "rubber band" whose tension $\tau = 1$ (regardless of the length of of c^*).*

The pressure competes with the tension: the pressure tries to collapse the curve c, while the tension tries to collapse the dual curve c^* in the opposite direction. To be a little

*We need not be concerned about practical aspects of building such a model; besides, a rigorous proof is given later.

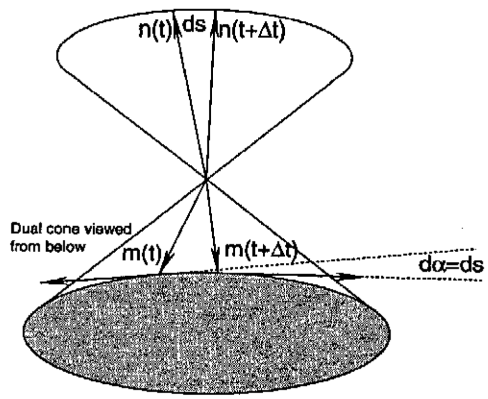


Figure 2.1. Proof of the dual cones theorem by mechanical analogy.

more precise, vectors n and n^* are being rotated around the $T \equiv n \times n^*$-axis in opposite directions by the pressure and the tension respectively. We claim:

> *The mechanical system just described is in neutral equilibrium for any* dual pair, c, c^*

Postponing the proof for a moment, we show how this statement implies the dual cones theorem. Let us deform the curve c to a point with the dual curve c^* becoming an equator. We spend zero work on such a deformation: $W = 0$, in view of the above statement. On the other hand, $W = W_1 + W_1^*$ where the work W_1^* stretches c^* from its starting length $L(c^*)$ to the length 2π of the equator, and of the work W_1 contracts the area A bounded by c to a point. Because of our choice of the tension and the pressure, $W_1^* = 2\pi - L(c^*)$ and $W_1 = -A(c)$.

Summarizing, we obtain

$$0 = W = 2\pi - L(c^*) - A(c)\,,$$

as the theorem claims, and it remains only to prove the equilibrium statement. To that end we take two dual elements ds and ds^* of c and c^* and show that the torque τ_1 created by the pressure on the element ds balances the torque τ_2 which is the resultant of the tensile forces on the endpoints of ds^*. Since $\dot{n}(t) \uparrow\uparrow \dot{n}^*(t)$, the torques τ_1 and τ_2 are collinear; in fact, they are oppositely directed* so that it remains to check that $|\tau_1| = |\tau_2|$.

*This latter fact is proven rigorously in sec. 2.2, although it is almost clear from figure 2.1.

We have $|\tau_1| = ds + O(ds^2)$. Considering now the projection of ds^* onto the plane tangent to the sphere at one end of ds^*, we obtain $|\tau_2| = |T_1 + T_2| + O((ds^*)^2)$, where T_1 and T_2 are the unit tangent vectors giving tension forces at the endpoints of ds^*, figure 2.1. Thus $|\tau_2| = \sin(d\alpha) + 0(d\alpha^2) = d\alpha + 0(d\alpha^2)$, where $d\alpha$ is the angle between the projections of T_1 and T_2 onto the tangent plane. On the other hand, $ds = d\alpha$, see figure 2.1, so that $|\tau_2| = ds + 0(ds^2)$ and thus the torques do balance. To summarize,

Boiled down to its essence, the proof of the dual cones theorem, and thus of the Gauss-Bonnet theorem, lies in the fact that the length ds of c measures the geodesic curvature $k^* ds^* = ds$ of c^*.

Remark 2.3. The above argument shows that $\frac{ds}{ds^*} = k_g(c^*)$, the geodesic curvature of c^*. Indeed, this follows from the definition $k_g(c^*) = \frac{d\alpha}{ds^*}$ and the remark that $\frac{d\alpha}{ds} = 1$.

2.2 The "moving fronts" proof of the dual cones theorem.

1. An outline of the proof goes like this. Let us deform the curve $c = c_0$ to a point curve c_1 through a one-parameter family of curves c_τ, $0 \le \tau \le 1$. The corresponding dual curves c_τ^* deform from $c_0^* = c^*$ to an equator c_1^*. Since the desired equality holds for $\tau = 1$: $A(c_1) + L(c_1^*) = 0 + 2\pi = 2\pi$, it suffices to show that

$$\frac{d}{d\tau}(A(c_\tau) + L(c_\tau^*)) = 0\,. \tag{2.2}$$

The curves c_τ and c_τ^* are traced out by unit vectors $n(t,\tau)$ and $m(t,\tau)$, $0 \le t \le 1$ respectively. Let $v(t,\tau)$ denote the normal velocity of the "moving front" c_τ at the point $n(t,\tau)$, and let $k_g(c_\tau^*)$ denote the geodesic curvature of the curve c_τ^* on the sphere. We have

$$\frac{d}{d\tau}A(c_\tau) = \int_0^1 v(t,\tau)|\dot{n}|\,dt\,; \tag{2.3}$$

indeed, the area bounded by a curve changes in proportion to the normal velocity of the moving boundary – this intuitively reasonable statement is proven rigorously below.

On the other hand, the *length* of a moving front changes in proportion to the product of its normal velocity and the geodesic curvature (again, a rigorous proof will follow):

$$\frac{d}{d\tau}L(c_\tau^*) = \int_0^1 v^*(t,\tau)k_g(c^*)|\dot{n}^*|\,dt\,. \tag{2.4}$$

Observing now that $v = -v^*$ – this is almost obvious from figure 2.1 – and that $k_g(c^*)|\dot{n}^*|dt = |\dot{n}|dt$ – this was explained in Remark 2.3 – we obtain the desired equality

$$\frac{d}{d\tau}A(c_\tau) - \frac{d}{d\tau}L(c_\tau).$$

2. We now fill in the technical details in the preceding proof. First we differentiate $A(c_\tau) = \int_0^\tau \int_0^1 n_\tau \times n_t \cdot n \, dt \, d\tau$ (cf. eq. (1.2)); here n_τ and n_t denote $\frac{\partial}{\partial \tau} n(t, \tau)$ and $\frac{\partial}{\partial t} n(t, \tau)$ respectively, and $\dot{n} \pm T|\dot{n}|$ is used:

$$\frac{d}{d\tau}A(c_\tau) = \int_0^1 n_\tau \times n_t \cdot n \, dt = \int_0^1 n_\tau \cdot (\dot{n} \times n) \, dt$$
$$= \pm \int_0^1 n_\tau \cdot (T \times n)|\dot{n}| \, dt = \pm \int_0^1 n_\tau \cdot n^* |\dot{n}| \, dt. \tag{2.6}$$

As a side remark we observe that $n_\tau \cdot (T \times n) = n_\tau \cdot n^* = v(t, \tau)$ is the normal velocity of the curve c (cf (2.3)). Denoting $n^* = m$ we differentiate the length $L(c_\tau^*) = \int_0^1 m \times \dot{m} \times T \, dt = \int_0^1 m \times \dot{m} \cdot n \, dt$:

$$\frac{d}{d\tau}L(c_\tau^*) = \int_0^1 m_\tau \times \dot{m} \cdot n \, dt + \int_0^1 m \times \dot{m}_\tau \cdot n \, dt + \int_0^1 m \times \dot{m} \cdot n_\tau \, dt.$$

The first term on the right-hand side vanishes since m_τ, $\dot{m}$ and n are all orthogonal to the same vector m. Similarly, the last term on the right-hand side vanishes because m, $\dot{m}$ and n_τ are perpendicular to n. Integration by parts of the surviving middle term gives

$$\frac{d}{d\tau}Lau = m \times m_\tau \cdot n|_0^1 - \int_0^1 \dot{m} \times m_\tau \cdot n \, dt - \int_0^1 m \times m_\tau \cdot \dot{n} \, dt$$
$$= \int_0^1 m \times m_\tau \cdot \dot{n} \, dt = \int_0^1 \dot{n} \times m \cdot m_\tau \, dt.$$

Substituting $\dot{n} = \pm T|\dot{n}|$ into the last integrand and recalling that $(-T) \times m = n$, we obtain

$$\frac{d}{d\tau}L = \pm \int_0^1 n \cdot m_\tau |\dot{n}| \, dt = \mp \int_0^1 n_\tau \cdot m |\dot{n}| \, dt;$$

the last step follows by differentiation of $n \cdot m = 0$ by τ, which gives $n_\tau \cdot m = -n \cdot m_\tau$. The last expression is the negative of (2.6) ◊

3. The Gauss-Bonnet theorem: proof via dual cones.

We recall the two-dimensional version of the Gauss-Bonnet theorem:

$$\int_\gamma k_g \, ds + \int_D K \, dS = 2\pi , \tag{3.1}$$

where D is a two-dimensional surface, topologically a disc, with the boundary curve γ (both γ and D are smooth*), k_g is the geodesic curvature of the curve γ, K is the Gaussian curvature of D and ds, dS are the arclength and the area elements of γ and D respectively.

3.1 Equivalence of the dual cones' theorem and the Gauss-Bonnet theorem.

Given a surface D with the boundary γ as described above, we will construct two associated cones C and C^*, show that they are dual and that the length and the area of these cones are precisely the integrals of the Gaussian curvature of D and of the geodesic curvature of γ respectively. A subsequent application of the dual cones' theorem will prove the Gauss-Bonnet theorem.

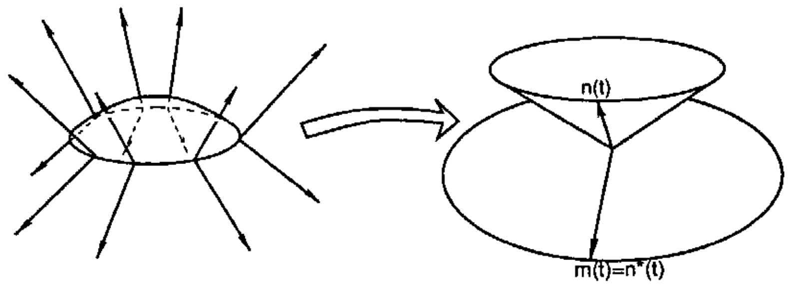
We introduce first some notations. Let γ: $[0, 1] \to \mathbb{R}^3, \gamma(0) = \gamma(1)$ be a parametrization of the boundary curve γ, and let $n(t)$ be the unit normal vector to the disk D at $\gamma(t)$ pointing in the positive direction according to a chosen orientation of the surface domain D. Let c be the image of γ under the Gauss map, i.e. let c be the curve on the unit sphere traced by the unit vectors $n(t)$.

We define the curve c^* by the set of unit vectors

$$m(t) = \frac{\dot{n}}{|\dot{n}|} \times n ; \tag{3.2}$$

interpreted physically, they give the direction of the angular velocity $\Omega(t) = n \times \dot{n}$ of the plane (or of the wheel) $\pi(t)$ tangent to the surface at the point $\gamma(t)$. The curve c^* thus defined is in fact dual to c, by the definition of duality. Incidentally, we assume here that $n \times \dot{n} \neq 0$ – this is the case if the Gaussian curvature $K \neq 0$; the general case can be treated by a limiting process.

*We look only at the case of smooth γ; one can allow corners as well, but doing so would not add new ideas to our discussion.



$$A(c) + L(c^*) = 2\pi \Leftrightarrow \int k_g + \int KdS = 2\pi$$

Figure 3.1. The definition of the Gauss cone C of normals and of the cone C^* of angular velocities of tangent planes, which is dual to C.

Remark. $A(C) = \int_D K\,dS$. Indeed, this is precisely the "integral of the definition" $dA = KdS$ of the Gaussian curvature as the determinant of the Gauss' map.

Lemma 3.1. $L(C^*) = \int_\gamma k_g\,ds$

We postpone the proof of this key lemma until the end of this section.

Proof of the Gauss-Bonnet theorem.

Combining the last three statements with the dual cones lemma, we obtain the Gauss-Bonnet theorem (3.1) at once, and it remains to prove lemma 3.1.

Proof of Lemma 3.1. The idea of the proof, see figure 3.2, is to look at $\beta' = (d/dt)\angle(T,m) = \omega(m) - \omega(T) = dl/dt - k_g$, where $\omega(T)$ is the angular velocity of T within the tangent plane to the surface. Integrating by t and using the periodicity of $\beta(t)$, we arrive at $\int k_g = L(C^*)$.

We proceed now with a more formal proof. Define the angle between two vectors v, w in the tangent plane π at a point on the surface D as $\angle(v,w) = sin^{-1}\frac{v}{|v|} \times \frac{w}{|w|} \cdot n$, where n is the unit normal vector to π in the positive direction (given by the orientation of the surface – in other words, the direction of n is chosen so that adding it to a positively oriented 2-frame in the tangent plane produces a positively oriented 3-frame in $\mathbb{R}^3$). For the future reference we note that if three vectors u, v, w lie in the same tangent plane, then $\angle(u,v) + \angle(v,w) = \angle(u,w)$.

As before, together with the normal vectors $n(t)$ along γ we introduce the unit vectors $m(t) = \frac{\dot{n}}{|\dot{n}|} \times n$ which give the directions of the angular velocity (along the spoke) of an imagined wheel whose axis is normal to the surface and which is not rotating around its axis. Lemma 3.1 says in effect that as we guide the wheel all the way around the loop γ, it will have turned by the amount equal to the integral of the geodesic curvature of γ. Let $\beta = \angle(T(t), m(t))$, and let $P_{\Delta t} v$ denote the parallel transport of a tangent vector v to the surface D along the curve γ from $\gamma(t)$ to $\gamma(t+\Delta t)$, figure 3.2.

We have, using the fact that $P_{\Delta t}$ is an isometry:

$$\begin{aligned}\beta(t) &= \angle(T(t), m(t)) = \angle(P_{\Delta t}T(t), P_{\Delta t}m(t)) \\ &= \angle(P_{\Delta t}T(t), T(t+\Delta t)) + \angle(T(t+\Delta t), m(t+\Delta t)) \\ &\quad + \angle(m(t+\Delta t), P_{\Delta t}m(t))\,; \end{aligned} \tag{3.3}$$

the last equality follows from the fact that all three vectors involved lie in the same plane $(\pi(t+\Delta t))$.

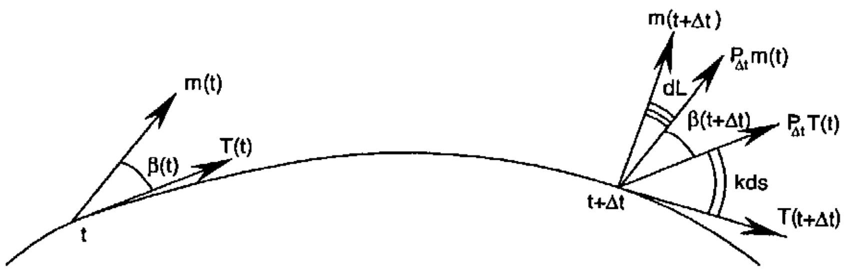


Figure 3.2 Proof of Lemma 3.1

The first term in the last sum is $k_g \Delta s + 0(\Delta s^2) \equiv k_g|\dot{\gamma}(t)|\Delta t + 0(\Delta t^2)$, by definition of k_g. The second term in (3.3) is $\beta(t+\Delta t)$, by definition of β. Finally, the last term is given by

$$\angle(m(t+\Delta t), P_{\Delta t}m(t)) = \sin^{-1}[m(t+\Delta t) \times P_{\Delta t}m(t) \cdot n(t+\Delta t)]\,, \tag{3.4}$$

where we have used the fact that the vectors have unit length, and that the direction of $n(t)$ is determined by the orientation of the tangent plane.

To find $P_{\Delta t}m(t)$ we recall that the parallel transport of a vector $v(t)$ with the tangent plane $\pi(t)$ can be defined by treating $v(t)$ as a spoke of a bicycle wheel lying in the plane $\pi(t)$ and having zero angular velocity along the normal direction $n(t)$; this causes $v(t)$ to move according to $v'(t) = \Omega(t) \times v(t)$, where $\Omega = \dot{n} \times n$ is the angular velocity vector. We have the rule of parallel transport: $P_{\Delta t}v = v + \Omega \times v\Delta t + 0(\Delta t^2)$. For $v = m(= \frac{\Omega}{|\Omega|})$ we obtain

$$P_{\Delta t}m(t) = m(t) + 0(\Delta t^2)\,,$$

which simply says that the points lying on the axis of rotation have zero velocity. Substituting this into (3.4) and using the Taylor expansions $m(t+\Delta t) = m(t)+\dot{m}(t)\Delta t+0(\Delta t^2)$ and $n(t+\Delta t) = n(t)+\dot{n}(t)\Delta t+0(\Delta t^2)$, we obtain, recalling that $|m| \equiv |n| \equiv 1$:

$$\begin{aligned}\angle(m(t+\Delta t), P_{\Delta t}m(t)) &= \sin^{-1}[\dot{m} \times m \cdot n\Delta t + 0(\Delta t^2)] \\ &= -\sin^{-1}\{m \times \dot{m} \cdot n\Delta t + 0(\Delta t^2)\}.\end{aligned}$$

Substituting this in (3.3) and dividing by Δt we obtain

$$\dot{\beta} + k_g|\dot{\gamma}| = m \times \dot{m} \cdot n\,,$$

as anticipated before, or integrating by t,

$$\int_\gamma k_g\, ds = \int_0^1 m \times \dot{m} \cdot n\, dt \equiv L(c^*)\,. \qquad \diamondsuit$$

4. Some properties of the dual curves and of Gauss' map.

We summarize here some interesting points that arose in the above discussion.

Theorem 4.1 *Total geodesic curvature $\int_\gamma k_g(\gamma)\, ds$ of a closed curve γ on a two-dimensional surface is an invariant of Gauss's map. In other words, the total geodesic curvature of γ equals the total geodesic curvature of the curve $c = G(\gamma)$ on the unit sphere given by unit normal vectors to the surface D along its boundary γ.*

Proof: Applying Lemma 3.1 to the boundary γ of D we obtain $\int_\gamma k_g(\gamma)\, ds = L(G(\gamma)^*)$. Application of the same lemma to the spherical curve $c = G(\gamma)$ gives $\int_{G(\gamma)} k_g(G(\gamma))\, ds = L(G(G(\gamma))^*)$. We complete the proof by observing that the Gauss map stabilizes under iterations: $G \circ G = G$.

Theorem 4.2 *Geodesic curvatures of dual curves c and c^* on the sphere are reciprocal at corresponding points:*

$$k_g(c)(t) \cdot k_g(c^*)(t) = 1\,.$$

Proof is given in Remark 2.2 at the end of sec. 2.1.

5. Other applications of the dual cones theorem: a "spherimeter", waveguides, twisted beams, ropes, ribbons and springs.

5.1 Measuring spherical areas.

The following mechanical device could be used to measure the area of a spherical domain D, figure 5.1. The device is just a small wheel, i.e. a gyroscope mounted at one tip O of an axis with the plane of the wheel perpendicular to the axis. The wheel encounters no rotational friction. The end O of the axis is held fast at the origin, while the other end E can be moved freely along the sphere. To measure the area $A(D)$ of a country, we place the endpoint E at some point on the boundary C of D; the wheel must be at rest. We then guide E along C bringing it back to the original point, and record the angle α by which the wheel has turned as the result of this circumnavigation. The area is given by

$$A = \alpha R^2 , \tag{5.1}$$

R being the radius of the sphere. Indeed, the wheel turns by $\alpha = 2\pi - L(c^*)$ since it rolls along the dual cone with no slipping (see Lemma 1.1 for more details). By the dual cones theorem $2\pi - L(c^*)$ is the solid angle described by wheel's axis; the corresponding spherical area is $(2\pi - L(c^*))R^2 = \alpha R^2$.

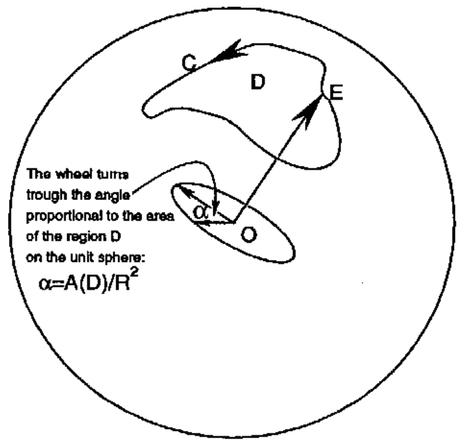


Figure 5.1 A spherimeter.

5.2. A geometrical model of twisted ribbons.

In this section we give a geometrical model which captures the intuitive concept of twisted one-dimensional objects such as thin rods or beams, ropes, hoses, etc.. We will refer to such objects collectively as *ribbons*. By a ribbon we understand a curve together with a normal bundle of unit vectors $e(s)$ orthogonal to the curve at each point. (here s measures the arclength along the curve).

Definition. The *twist* τ of the ribbon is the angular velocity (with s treated as time) of the normal frame $(e(s), e(s) \times T(s))$ in the direction of the tangent vector $T(s)$ to the curve. Analytically, $\tau = \dot{e} \cdot e \times T$. In the particular case when $\tau = 0$, i.e. when the frame is inertial, we will refer to the ribbon as *untwisted*, or *inertial*.

We consider a ribbon with two endpoints having parallel tangent vectors; this is the standing assumption, unless stated otherwise. There is a simple relationship between the total twist $\int_0^L \tau \, ds$, the angle α through by which the normal frame $(e_1(0), \; e_2(0)) = (e(0), \; e(0) \times T(0))$ is rotated into the frame $e_1(L), \; e_2(L)$ and the shape of the ribbon:

Theorem 5.1. *For a ribbon with parallel ends, and with the twist τ, the angle α between the normal vectors $e(0)$ and $e(L)$ is given by*

$$\alpha = A + \int_0^L \tau \, ds \,, \tag{5.2}$$

where A is the solid angle subtended by the cone of unit tangent vectors (carried over to the origin) to the curve. In particular, for the case of an inertial ribbon: $\tau = 0$, we have $\alpha = A$. Here α and A are defined with respect to the orientation given by the tangent vector T (when T points into the observer's eye, the counterclockwise rotation is the positive one).

Proof. This theorem follows at once from our observations on the bicycle wheel. First we take the case of $\tau = 0$ of the untwisted ribbon. Let us imagine that a wheel is transported along the curve so that (i) its axis is tangent to the curve and (ii) the angular velocity of the wheel in the axial direction is zero. Then one of the spokes of the wheel gives the inertial frame $e(s)$ of the ribbon. Let C be the cone swept out by the wheel's positive semiaxis (transported to the origin), and let C^* be its dual cone. Lemma 1.1 on the disc rolling on a cone gives $\alpha = 2\pi - L(C^*)$, and by the dual cones' theorem, $\alpha = A$, proving the case of $\tau = 0$. In the general case, τ has the meaning of the angular velocity in the tangent direction, and thus its cumulative contribution results in the integral in eq. (5.2). ◇

Corollary 5.1. *Consider an untwisted ribbon ($\tau = 0$) forming a* closed loop, *and let us subject the ribbon to a deformation which maintains $\tau = 0$ while preserving the length of the curve. There is an interesting rigidity imposed by the zero twist condition: any such deformation of the ribbon preserves the area A of the hodograph, or the Gauss image of* *the curve, i.e. the area of the region on the unit sphere enclosed by the unit vectors tangent to the curve *. In particular, the hodograph image of any twist preserving deformation of a closed untwisted ribbon whose base curve is initially planar divides the the area of the sphere precisely in half.*

This corollary can be demonstrated by taking a piece of a garden hose, closing it up into a planar ring by tying tightly the two ends. The hose can be thought of as a ribbon, and the deformation of the hose preserves zero twist: it is much easier to bend the hose than to twist it. This zero twist condition imposes the constraint on the possible shapes of the bent hose: the hodograph image of the curve formed by the hose equipartitions the area of the sphere.

Corollary 5.2. *If two ends of a straight ribbon with zero twist are twisted relative to each other through an angle α, with the deformed ribbon still satisfying the zero twist condition, then the base curve of the ribbon compensates for this global twist by the buckling whose amount is measured by the solid angle A of its hodograph image.*

5.3 Waveguides

When viewing an object through an optical fiber, we may observe that the image is at an angle to the object itself, Figure 5.2. What is this angle α? Again, the answer turns out to be $\alpha = A$, where A is the solid angle of the cone of tangent vectors to the curve carried to the origin.

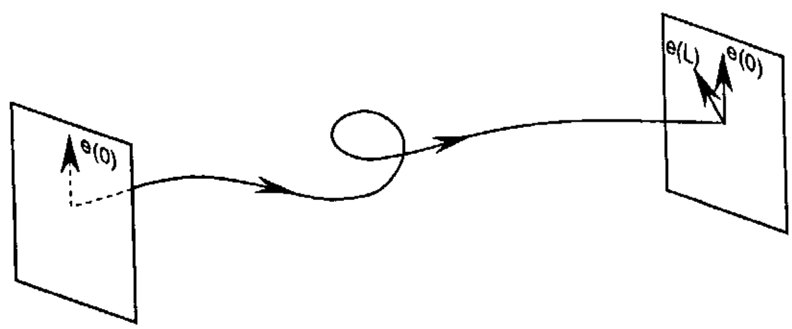


Figure 5.2 The tangent vectors at the endpoints of the optical fiber are assumed parallel. The angle α between the object and its image is given by the solid angle A.

Indeed, consider a one-dimensional carrier of transversal waves such as an optical fiber. We model this object by a curve in $\mathbb{R}^3$ with two endpoints, and consider a transversal

*After these vectors have been carried to the origin.

polarized wave propagating along the curve; the direction of the polarization is given by a family of vectors $e(s)$ normal to the curve at every point. We assume that the wave propagates in such a way that the polarization bundle $e(s)$ is *inertial*, i.e. the angular velocity vector of the frame $(n, e, n \times e)$, where n is the tangent vector to the curve, is *orthogonal* to n. Equivalently we can think of transporting a wheel (whose axis is tangent to the curve) along the curve while maintaining zero angular velocity around its axis at all times. For the case of the optical fiber the wheel gives the connection, i.e. the rule according to which the two-dimensional image is carried along the fiber from one transversal section to another. We assume now that the tangents to the two ends of a fiber are parallel, so that it makes sense to talk about the angle α between the object lying in the plane P_{object} and its image lying in the parallel plane P_{image}.

Again, we can calculate α by considering the cone C of tangent vectors to the curve and the dual cone C^*, the envelope of normal planes to the curve, with all planes and lines translated to the origin. The key observation is that the image-carrying plane rolls without sliding on the cone dual to the cone C of tangent vectors to the curve, so that $\alpha = 2\pi - L(C^*) = A(C)$. This can be stated as

Theorem 5.2. *The Hannay-Berry angle α of the waveguide described above is given by the solid angle of the cone of tangents to the waveguide.*

An example: a helical waveguide. We consider one loop of a helix given by

$$r(t) = (R\cos t, R\sin t, at)\,.$$

The unit tangent vector

$$T = \frac{1}{|\dot{r}|}\dot{r} = \frac{(-R\sin t, R\cos t, a)}{\sqrt{R^2 + a^2}}$$

traces out a circle on the unit sphere which encloses a spherical cap; the area A of any spherical cap equals 2π times its height, which gives $A = 2\pi(1 - \frac{a}{\sqrt{R^2+a^2}})$. Since $\alpha = A$, we have

$$\alpha = 2\pi\left(1 - \frac{1}{\sqrt{1 + (\frac{R}{a})^2}}\right). \tag{5.3}$$

For small $\frac{R}{a} \approx 0$ the helix is close to a straight line and consequently α is small, an obvious result. Fixing R and decreasing a towards zero, we make the spiral tight and thus $\alpha = 2\pi\left(1 - \frac{1}{\sqrt{1+(\frac{R}{a})^2}}\right) \approx 2\pi$. We conclude that *straightening a near circular loop of the helical waveguide causes the image to turn by (nearly)* 2π. This has another interpretation in terms of twisted ribbons, which we describe now.

5.4 Twisted beams, ropes, springs and ribbons.

If one holds a straight beam by its ends and applies opposite torques to its ends, trying to twist the end sections of the beam in their planes in the opposite directions, the beam would buckle. The same phenomenon is observed when a rubber band develops helical structure when twisted. Another manifestation of this effect is observed when a rope or a garden hose is removed from a spool by slipping it off the fixed spool sideways rather than by rotating the spool-the ropes and hoses acquire a twist; if one tries to get rid of this twist one gets coiling instead. This annoying phenomenon can be explained and quantified via the Gauss-Bonnet theorem, or the dual cones theorem. Mathematically the model of the twist in the rope or the beam is the same as the above model of a polarized wave travelling through a waveguide. On the other hand, the reason why a spring has elasticity is again explained by the same mechanism: stretching of the spring translates into the *twisting* of the spring's wire; it is precisely this twist that shows up in the desire of the spring to resist stretching or compression. The relationship between the twist and the stretching is quantified by theorem 5.2 proceed now to show.

Springs.

Why are springs elastic? How to calculate the coefficient of elasticity of the spring? These questions can be answered using eq. (5.2). Let us stretch a spring, while keeping its helical shape, and so as not to change the angle between the end crosssections. Which we assume the spring to consist of an integer number of loops. This condition amounts to keeping $\alpha = \text{const}$. Applying eq. (5.2) to one loop of the helix, we obtain $\tau = (\alpha - A)/L$, L being the length of the loop. Let A_0 be the solid angle for the unstressed spring, i.e. for $\tau = 0$; in this case eq. (5.2) gives $\alpha = A_0$, so that

$$\tau = (A_0 - A)/L\,.$$

Now, the geometrical twist τ determines the twisting torque T acting on the springs's wire; this stress-strain relationship between T and τ depends on the elastic properties of the material (for small deformation it can be assumed linear). Once T is known, it is a simple geometrical exercise to determine the pulling force on the spring caused by the torque T.

Interpretation of the writhing number via dual cones and Berry's phase.

Berry's phase, of which the solid angle A in the above discussions is a principal example, is closely related to the writhing number which arises in the geometrical studies of the DNA as well as in some aspects of wave propagation. I noticed this relationship thanks to Steve Strogatz, who showed me his paper [TS] with J. Tyson and who independently came up with the observations made below; the reader is referred to this paper for further references and for a very nice survey of the subject.

The writhing number is defined as

$$W_{\tau} = SL - Tw\,,$$

where SL is the integer self-linking number, i.e. the linking number of the two curves, one being the base curve and the other given by shifting the base curve along the direction of $e(s)$ by a small amount, and $TW = \frac{1}{2\pi}\int\tau$ over the curve. To establish the relationship between Berry's phase and the writhing number, we change the ribbon by keeping the given curve but changing the frame; since Wr depends only on the curve ([F]), this is an allowed operation. Let us take a frame which is inertial except at one point, where it has a discontinuity; the latter is precisely the Berry's phase α. We smooth this discontinuity, obtaining the ribbon with $\int\tau = \alpha$, whose twist therefore is

$$Tw = \frac{1}{2\pi}\alpha\,,$$

everything is measured with respect to the orientation of the curve. Substituting this into the definition of Wr, we obtain $Wr = SL - \frac{1}{2\pi}\alpha$, or

$$\alpha = 2\pi(SL - Wr)$$

This relationship shows that, up to a sign and up to an integer and up to a multiple of 2π, the writhing number is the same as Berry's phase. This allows one to interpret the writhing number in terms of cones and dual cones. In particular, this allows an (apparently new) interpretation of Wr in terms of the "length" of dual cones: $Wr = \frac{1}{2\pi}L(C^*)(\mathrm{mod}\,1)$, where the cone C^* is the envelope of the normal planes to the curve, carried to the origin.

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