A Coriolis Pair Paradox

MATHEMATICAL CURIOSITIES
By Mark Levi

I am walking the straight chalk line AB drawn through the center of a rotating platform. Although my path is straight and my speed is constant in the platform's frame, neither is true in the inertial frame of the ground observer. The nonzero acceleration at the center O in Figure 1 is precisely the Coriolis acceleration.[1] Its magnitude is

$$a_{\text{coriolis}} = 2v\omega, \qquad (1)$$

where v is the speed relative to the disk (which at O is the same as the ground speed).

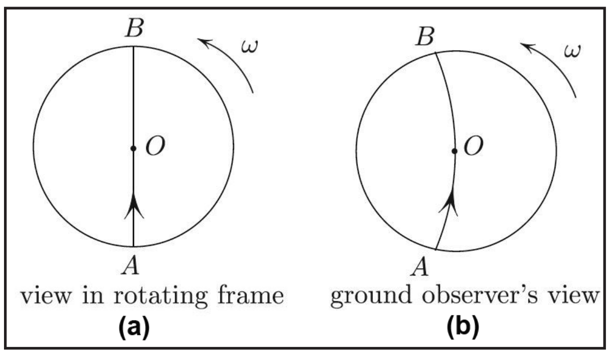

Figure 1. *Coriolis acceleration at O.*

A Side Note

There is some ambiguity in the Coriolis terminology, similar to that between centrifugal and centripetal forces. Coriolis force sometimes refers to the inertial (i.e., fictitious) force that is equal and opposite to the aforementioned force. The walker in Figure 1a who cannot see outside the carousel, feels as if some force is pulling him to the *right* — just like a passenger in an accelerating car feels a backward pull. Since nothing actually pulls them backwards, one speaks of inertial, fictitious (D'Alembert's) force. As a more formal example, the position vector $\mathrm{x} \in \mathbb{R}^2$ of a free particle that is expressed in the rotating frame does not satisfy $\ddot{\mathrm{x}} = 0$ but rather $\ddot{\mathrm{x}} = -2i\omega\dot{\mathrm{x}} + \omega^2\mathrm{x}$, using the complex notation, i.e., identifying $\mathrm{x} = (x, y) \equiv x + iy$.

[1] The centripetal force vanishes at O, but away from O the Coriolis force is only one part of the inertial force — specifically the part due to the motion that is relative to the platform. The other part is the centripetal force, which points to the center and prevents the walker from flying uncontrollably outwards, only depends only on the position (not on the velocity) [1, 2].

The Paradox

Figure 2 illustrates a back-of-the envelope attempt to derive (1) by computing the change of speed in the x-direction relative to the ground in a short time from $t = 0$ to $t > 0$. At time $t > 0$, the walker has traveled the distance $r = vt$ from O; the platform's rotation imparts speed $\omega r = \omega vt$ to the walker. The x-component is $\omega r \cos\omega t = \omega vt \cos\omega t$, and the x-acceleration at O is therefore

$$\lim_{t\to 0} \frac{\omega vt \cos\omega t - 0}{t} = \omega v.$$

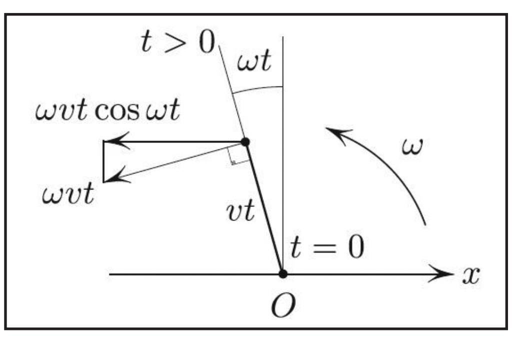

Figure 2. *Speed change in the x-direction that is caused by the platform's rotation. The x-axis is affixed to the ground.*

But this is only half of the correct answer (1). Where is the other half?

The Missing Half

Walking on the platform has *two* effects, and I overlooked one: At $t > 0$, my velocity relative to the ground has rotated together with the disk, thus contributing the missing ingredient to the x-speed. Figure 3 shows this overlooked contribution as $v \sin\omega t$. The missing part of the Coriolis acceleration is thus

$$\lim_{t\to 0} \frac{v \sin\omega t}{t} = v\omega.$$

Interestingly, the two contributions end up being equal, but with a subtle difference: ωv and $v\omega$. Figure 4 shows the two effects together.

Coriolis Force in a Jetliner

What is the Coriolis force acting on a jet that is flying over the North Pole? For a Boeing 747, let us take $v = 250\ m/sec$, $m = 400 \cdot 10^3\ kg$. Substituting Earth's angular velocity $\omega = 2\pi/(24 \cdot 3,600)\ rad/sec$ into $F = 2m\omega v$ yields a Coriolis force of roughly 3,000 pounds. It is as if the plane were pulled to the right with a force that is equal to the weight of a small elephant. I never expected such a large Coriolis force and recalculated it several times, anticipating a different result each time and thinking of the famous quote: "*Insanity is doing the same thing over and over again and expecting different results*."

The figures in this article were provided by the author.

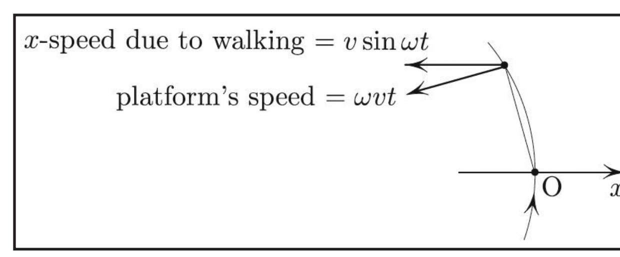

Figure 4. *The pair of ingredients (from the ground observer's view), each producing acceleration $v\omega = \omega v$ at O and resulting in the factor 2 in (1).*

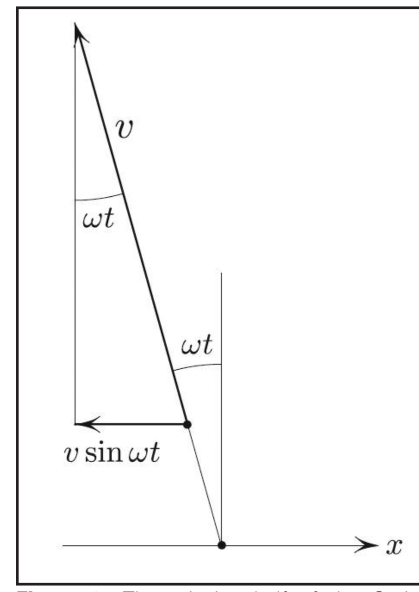

Figure 3. *The missing half of the Coriolis acceleration (from the ground observer's view).*

References

[1] Arnold, V.I. (1989). *Mathematical Methods of Classical Mechanics*. New York, NY: Springer-Verlag.

[2] Levi, M. (2014). *Classical Mechanics with Calculus of Variations and Optimal Control: An Intuitive Introduction*. In *Student Mathematical Library* (Vol. 69). Providence, RI: American Mathematical Society.

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