Strolling Through Jacobi Fields

MATHEMATICAL CURIOSITIES

By Mark Levi

What is a Jacobi Field?

A Jacobi vector field governs the separation of two nearby geodesics, to the leading order; the Jacobi equation is the linearization of the geodesic equation around a geodesic. In mechanical terms (and for embedded surfaces in $\mathbb{R}^3$), the distance s between two point masses that are sliding abreast on a surface (see Figure 1) in the absence of gravity and friction satisfies *to the leading order* the Jacobi equation

$$\ddot{s} + K(r)v^2 s = 0; \qquad (1)$$

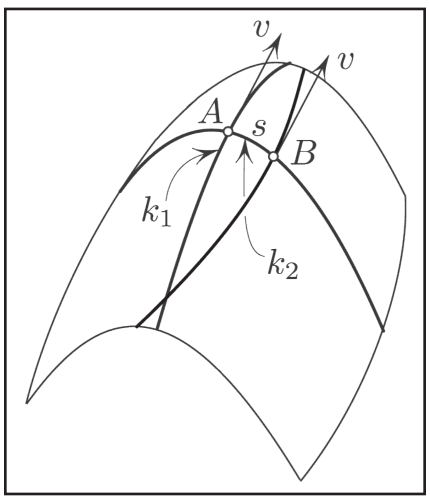

Figure 1. *The geodesic at A is tangent to a principal line of curvature of normal curvature k_1 (not drawn).*

here, $r = r(t)$ is the position of one of the masses, $v = |\dot{r}|$ is the speed, and K is the Gaussian curvature. Most books on differential geometry derive (1), but the derivations require some background as well as some space and some time. Instead, I would like to give a back-of-the-envelope derivation of (1) *in a special case* using little more than the high school formula $F = mv^2/R$ for the centripetal force.

The Setup

Consider a unit point mass A with a velocity that points in the direction of a principal curvature at the instant in question; an identical particle B is near to and abreast of A (i.e., the arc AB is perpendicular to the velocity of A). Both A and B have the same constant speed v, and B's direction of motion is close to that of A by assumption (see Figure 1). As mentioned, there is no gravity or friction.

A Heuristic Derivation of (1)

An observer who is sliding with the reference frame of A feels the centrifugal g-force due to the curvature of A's path:

$$F = k_1 v^2. \qquad (2)$$

This inertial force—which also acts on B from A's point of view—has the tangential component $F_{\text{restoring}} = F \sin\theta$ that pulls B towards A (see Figure 2). But $\theta = k_2 s + o(s)$ by the definition of curvature (see Figure 3). Therefore,

$$F_{\text{restoring}} = F k_2 s + o(s) \overset{(2)}{=} k_1 k_2 v^2 s + o(s).$$

And since $k_1 k_2 = K$ is the Gaussian curvature, this explains (1) — but only for the special case when the velocity points in a principal direction of curvature.

How to Explain (1) Heuristically for an Arbitrary Direction?

I would like to leave this question as a fun problem and may address it in the next installment. It turns out that the special case that I describe here misses an interesting aspect, one that is also hidden in the formal machinery of standard derivation. For example, what if the geodesic is a straight line on a ruled surface? Considering this question yields a mechanical interpretation of the Hessian determinant that has not occurred to me before.

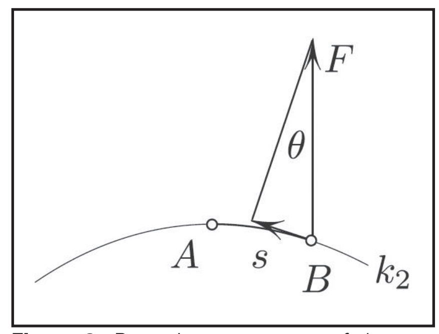

Figure 2. *Restoring component of the centrifugal force. Here, s is the arc length.*

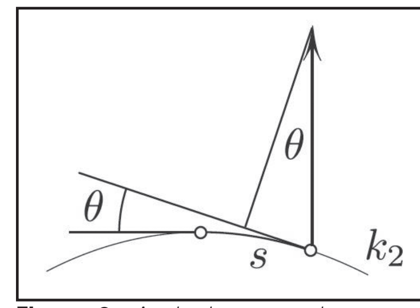

Figure 3. *Angle between the normals equals the angle between the tangents; the latter $\approx k_2 s$.*

The figures in this article were provided by the author.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.