A Bike and a Catenary

MATHEMATICAL CURIOSITIES
By Mark Levi

There is a surprising connection between the catenary (the shape of the hanging chain, given by the hyperbolic cosine) on the one hand, and the pursuit curve, also known as the tractrix (as illustrated in Figure 1) on the other. The tractrix is defined by the property that every tangent segment *RF* to a given line *MN* has fixed length; it is the track of the bike's rear wheel *R* when its front wheel *F* follows a straight line. Our bike is just a moving segment *RF* of *fixed length*, which we take to be 1, with the velocity of *R* constrained to the line *RF*.

Figure 2 summarizes the connection: all normals to the tractrix are tangent to the catenary (so that the tractrix is the *involute* of the catenary). Equivalently, if we let a string ATR_0 hug the catenary and—keeping the end *A* fixed—unwind the end *R* while holding the string taut, *R* will sweep a tractrix.

Yet another way to put it: as the bike in Figure 1 moves as shown, the line of its rear axle remains tangent to the catenary. And the tangency point *T* is the center of curvature of the bike's rear track.

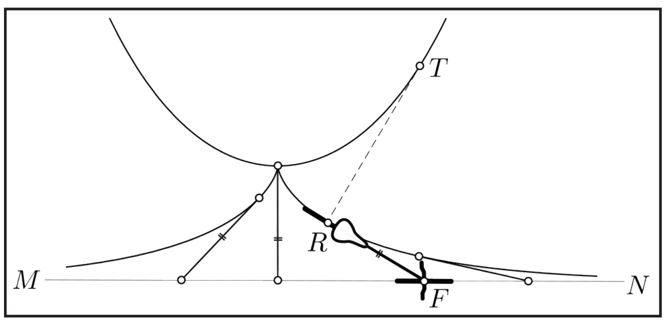

Figure 1. *The catenary and the tractrix.*

To prove this connection, consider an arbitrary position of the "bike" *RF* in Figure 2, and let *T* be the point of intersection of the normal through *R* and the normal to *MN* at *F*. The point *T* is automatically the center of curvature of the tractrix at *F*; leaving out the proof of this fact (which I will address in next month's column), we show that *T* traces out a catenary, i.e., that $y=FT=\cosh x$, where *x* is the coordinate of *F* on the line. From Figure 3,

$$y=\frac{1}{\sin\theta}. \qquad (1)$$

According to Figure 4,

$$\theta'=\frac{d\theta}{dx}=-\sin\theta. \qquad (2)$$

Indeed, the angular velocity of the "bike" is given by the difference of the sideways velocities of *F* and *R* divided by the length $|RF|=1$; θ decreases in the figure, which explains the minus sign. Differentiating (1) and using (2) we get

$$y'=\frac{dy}{dx}=\cot\theta.$$

One more differentiation and one more use of (2) gives

$$y''=-\frac{1}{\sin^2\theta}(-\sin\theta)=\frac{1}{\sin\theta}=y,$$

so that *y* is a combination of $\cosh x$ and $\sinh x$. And since $y(0)=1$ and $y'(0)=0$, we conclude that $y=\cosh x$, the equation of a catenary, as claimed.

In conclusion, here is an intriguing consequence of the catenary-tractrix connection. Consider the two surfaces of revolution generated by spinning each curve around the line *MN* in Figure 1. The surface of revolution of the catenary has zero mean curvature (it is the shape of a soap film spanning two circular hoops), while the surface of revolution of the tractrix has constant Gaussian curvature −1 (a pseudosphere). Is this just a coincidence, or a sign of something deeper?

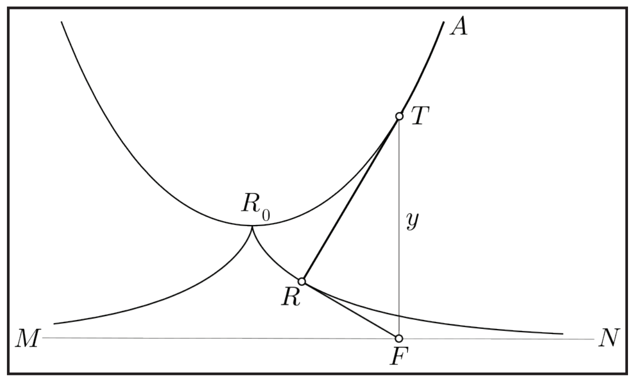

Figure 2. *As the string unwraps from the catenary, its end R describes the tractrix.*

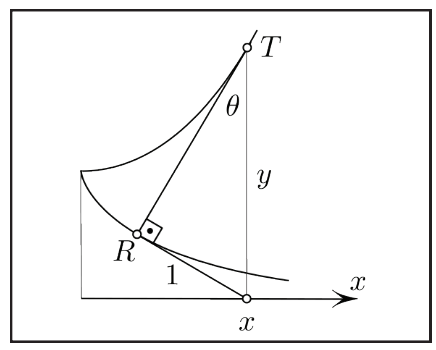

Figure 3. *Explanation of (1).*

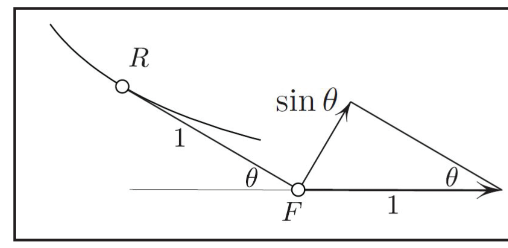

Figure 4. *Explanation of (2).*

The figures in this article were provided by the author.

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