Dynamics of Discrete Frenkel–Kontorova Models

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1. Introduction.

In this note we will describe an interesting class of waves propagating through a discrete medium. The prime example is a lattice of infinitely many coupled pendula given by the system

$$L\phi_n + k(2\phi_n - \phi_{n-1} - \phi_{n+1}) = I, \tag{1.1}$$

where $L \equiv \ddot{\phi} + \gamma\dot{\phi} + \sin\phi$, and $n \in \mathbf{Z}$, with constant parameters γ, k and I. This system describes an infinite chain of pendula with nearest neighbor coupling, in a periodic potential and in the presence of forcing and damping. Another physical interpretation of eq. (1.1) is sketched in Figure 1.1.

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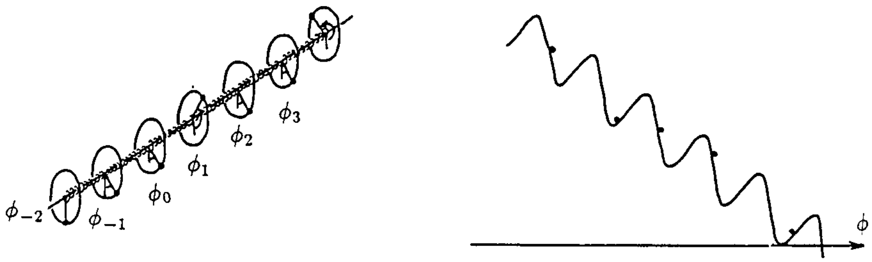

Figure 1.1. Eq. (1.1) describes an infinite chain of particles in a slanted potential $-\cos\phi - I\phi$ with the nearest–neighbor elastic coupling in the presence of damping $\gamma\dot{\phi}$

This system can be written formally in a more compact form

$$\ddot{\phi} + \gamma\dot{\phi} + \frac{\partial V}{\partial \phi} = 0,$$

where $\phi = (..., \phi_{-1}, \phi^0, \phi_1, ...)$ and the potential is the formal expression

$$V = \sum_{\infty}^{\infty} \frac{1}{2}(\phi_{n+1} - \phi_n)^2 + \cos\phi_n - I\phi_n.$$

All the results and proofs of this note carry over almost verbatim to the case of a more general nonlinear coupling, forcing and a not necessarily sinusoidal potential. We have chosen a specific case of eq.(1.1) because it really contains the substance of the more general problem

$$\ddot{\phi}_n + f(\dot{\phi}_n) + p(\phi_n) + g(\phi_n - \phi_{n-1}) + g(\phi_n - \phi_{n+1}) = I, \qquad (1.2)$$

where the generalized damping f and coupling g are nonlinear functions satisfying some mild monotonicity and growth assumptions, and where p is a periodic function satisfying some mild nondegeneracy assumptions. There is a robustness of behavior in such systems that is in contrast with the more delicate situation in completely integrable systems such as the so–called sine–Gordon equation $\phi_{tt} - \phi_{xx} + \sin\phi = 0$ with no damping and no forcing.

Eq. (1.1) arises in several mathematical and physical contexts, among which are the following.

1. Eq. (1.1) is a discretization of the damped driven sine–Gordon equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = I, \tag{1.3}$$

with ϕ_{xx} replaced by the second difference. The dissipation $\gamma\dot{\phi}$ and the forcing I destroy the complete integrability of the system, producing the new effects and the need for different methods.

2. Eq. (1.1) is a dynamical version with dissipation of the Frenkel–Kontorova model [8] arising in the solid–state physics and used by Aubry in his work [1].

3. Arrays of Josephson junctions (SQUIDS, or superconducting quantum interference devices) coupled in a nearest neighbor way are described by eq. (1.1).

4. Systems of coupled pendula provide a phenomenological model for charge–density waves (CDW) in anisotropic crystals [9].

A particular case of two particles:

$$\ddot{\phi}_1 + \gamma\dot{\phi}_1 + \sin\phi_1 + k(\phi_1 - \phi_2) = I_1,$$
$$\ddot{\phi}_2 + \gamma\dot{\phi}_2 + \sin\phi_2 + k(\phi_2 - \phi_1) = I_2,$$

has been analyzed in [12], [15], [16], [28] for small k and in [4], [11] for larger k.

The main result of this note is the analysis of a discrete analog of solitary traveling waves for eq. (1.1). These solutions, which exist for certain parameter values we specify later, behave as follows. At $t = 0$ all ϕ_j, $j \le 0$ lie in the interval $[0, \frac{\pi}{2}]$, while the remaining ϕ_j lie in the interval $[2\pi g,\ 2\pi g + \frac{\pi}{2}]$, where g is an integer depending on the parameters in the equation. As t grows, ϕ_0 leaves the left interval and travels to the right, entering the right interval at some $t = O(k^{-1})$; the other particles stay meanwhile in their respective intervals. After ϕ_0 has compits trip, ϕ_1 repeats the same procedure, etc..

Together with these solutions, for which one particle at a time makes a trip, there exist countably many other solutions behaving in the same way. These solutions are obtained from the ones described above by separating ϕ_j by adding a fixed multiple of 2π to all the distances between every two neighbors, figure 1.1. One observes the gap traveling to the left as the particles travel to the right; this is a discrete version of a rarefaction wave. Such solutions resemble an effect similar to slipping and pinning in charge–density waves [9].

All the results mentioned here hold for a periodic chain as well, figure 1.1. Such solutions exist provided the parameters I and k lie in the regions of the (I, k)–plane described in the theorem in the next section, Figure 2.1.

The precise statements are given in the next section.

Remark: discrete vs. continuous model.

Let h be the mesh size on the x–axis, and let $k = \frac{1}{h^2}$ and $\phi_n = \phi(nh)$. Then eq. (1.1) is the second difference approximation of (1.3), and for large k the approximation is good; for small k it is bad. Our methods in this note do not extend to the case of large k. There is, in fact, a strong qualitatitative dissimilarity between the discrete and the continuous equations when k is small. For instance, the discrete model (1.1) possesses an extermely rich set of equilibria; studying these reduces to the study of area–preserving cylinder maps; our understanding such maps is by no means complete: KAM and Aubry–Mather theories still leave many questions unaswered. As k in eq. (1.1) grows, this rich structure "melts", until at most two equilibria remain in the limit (1.3) of $k \to \infty$ (all this provided $I \neq 0$). These surviving equilibria are the two stationary solutions of eq. (1.3) constant in both x and t (they exist only when $|I| < 1$). This simplification of the behavior is due to the fact that an autonomous flow in the plane is a much simpler object than a mapping of the plane.

2. Traveling waves.

2.1. Symmetries.

Before stating the main result, we point out the symmetries in the problem. Let $\sigma\phi$ denote the right shift of the sequence $\phi = (\ldots\phi_{-1}, \hat{\phi}_0, \phi_1, \ldots)$ defined by

$$(\sigma\phi)_j = \phi_{j-1}, \tag{2.1}$$

let τ_n be the translation of the vector ϕ defined by

$$(\tau_n\phi)_j = \phi_j + 2\pi n, \tag{2.2}$$

let $\delta\phi$ be a discrete dilation of ϕ defined as

$$(\delta\phi)_j = \phi_j + 2\pi j, \tag{2.3}$$

and let ρ be the reflection in the index:

$$(\rho\phi)_j = \phi_{-j}. \tag{2.4}$$

Lemma 2.1. *Solution set of eq. (1.1) is invariant under σ, τ, δ and ρ.*

Proof is obvious.

2.2. The main theorem.

Theorem 2.1. *Fix the dissipation constant $\gamma > 0$ and an integer period $N > 1$ of the chain. There exists a set of open overlapping domains D_g in the (I, k)-plane, enumerated by all positive integers $g \geq g_0 = g_0(\gamma)$, figure 2.1. such that for all $(I, k) \in D_g$ there exists a periodic "traveling wave" solution $\phi(t) = (...\phi_{-1}, \phi_0, \phi_1, ...)$ behaving as follows.*

The solution has an integer spatial period N:

$$\phi_{j+N} = \phi_j + 2\pi g. \tag{2.5}$$

We allow $N = \infty$, in which case there are just two clusters, "left" and "right". At $t = 0$ all ϕ_j lie in clusters of N particles each, in the intervals $J_n = [0, \frac{\pi}{2}] + 2\pi gn$ – more precisely, $\phi_1(0), ..., \phi_N(0) \in J_0$, and $\phi_{j+nN}(0) \in J_{nN}$ for $j = 1, 2, ..., N$ and $n = 0, \pm 1, \pm 2, ...$

As t increases from 0, the last members $\phi_{nN}(t)$ from each cluster in J_n travel simultateously to the next cluster in J_{n+1} while the remaining particles are confined to their intervals J_n. After $\phi_{nN}(t)$ enter the neighboring interval J_{1+n} to the right, the left neighbors ϕ_{nN-1} repeat exactly the same process; after N trips all the ϕ's from the interval J_n enter the following interval J_{n+1}. The corresponding solution is time-periodic modulo index shifts σ: there exists $T > 0$ such that

$$\sigma\phi(t + T) = \phi(t), \tag{2.6}$$

and in the case $N < \infty$

$$\phi(t + NT) = \tau^g \phi(t), \tag{2.7}$$

i.e. in time NT all ϕ's are shifted by $2\pi g$.

There exists, moreover, another unstable periodic solution behaving as described above, except the intervals J_n should be replaced by $J_n + \frac{\pi}{2} = [\pi n g + \frac{\pi}{2}, \pi(n+1)g]$.

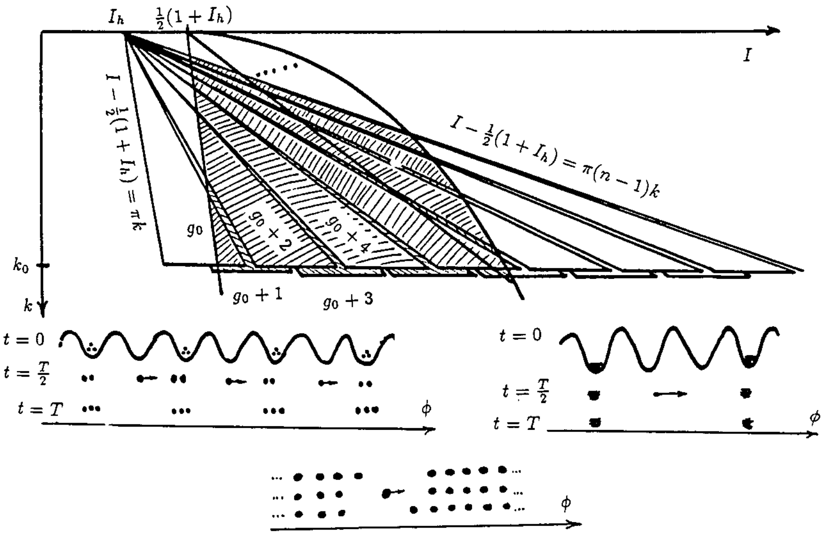

Figure 2.1. Domains D_g and some lattice waves for $N \geq 2$, $N = \infty$. The integer label g of the domain D_g gives the gap size of the traveling wave.

Discussion.

1. The stable and the unstable solutions mentioned in the theorem undergo a saddle–node bifurcation, in the topological sense. It has not been proven, although it seems almost certain, that as the point (I, k) crosses from one region D_g into D_{g+2}, the pair of solutions disappears in one saddle–node bifurcation shortly after D_g is left and a new pair is born shortly before D_{g+2} is entered – we point out that the domains D_g constructed in the proof are not the best possible and could be extended somewhat from the estimates given below.

2. By choosing $N = \infty$ we obtain only two clusters in J_0 and in J_1, and one particle at a time makes a trip from J_0 to J_1, as mentioned already in the introduction.

3. Symmetries (2.1)–(2.4) produce countably many other periodic solutions out of each of the two periodic solutions described in the statement of the theorem. The dilational symmetry (2.3), for instantce, produces additional periodic solutions by spreading the particles by a fixed integer multiple of 2π; in particular, the particles from one cluster will be spaced apart. The resulting configuration is shown in figure 3.1.

4. The repetition of the profile at time intervals T is the discrete version of the constancy of profile of a traveling wave for eq. (1.3), which can be expressed by the relation $u(x,t) = U(x - vt)$, or equivalently, by $u(x, t + \sigma) = u(x - v\sigma, t)$ for any $\sigma \in bfR$, reflecting the equivalence of the actions of the groups of time and space translations. This is the precise analog of eq. (2.6) for the discrete system. Since in our problem the space translations form a discrete group, so do the time translations.

5. The overlapping nature of the domains D_g implies the coexistence of two traveling waves with distinct gap sizes, g and $g + 1$. This peculiar phenomenon is not present in the countinuous case of the sine–Gordon equation and is explained in section 3 where the integer g is defined. We can also point out that the unstable solution corresponds to the antikinks.

3. Proof of the theorem.

3.1. A heuristic discussion.
3.2. An outline of the proof.
3.3. Two auxiliary systems.
3.4. Definition of the gap length g.
3.5. Construction of the domains D_g.
3.6. Poincaré sections Σ, Σ'.
3.7. Poincaré mapping $F : \Sigma \to \Sigma'$ is into.

3.1. A heuristic discussion.

Discrete traveling waves described here have the same underlying mechanism as the "caterpillar" solutions studied in [16]. We adapt the heuristic discussion from that paper to the system at hand. The dynamical explanation is the same as for the case of two pendula, but there are interesting complications of algebraic nature, which give rise to the phenomenon of coexisting stable traveling waves. To be specific, we concentrate on the case $N = \infty$ which corresponds to two clusters of particles with $..., \phi_{-1}, \phi_0$ lying in the interval $J_0 = [0, \frac{\pi}{2}]$ and $\phi_1, \phi_2, ...$ lying in $J_1 = 2\pi g + J_0$; in the present discussion we will give a rough estimate of the integer g and of the parameters γ, I, k needed for the traveling waves described here to occur.

Rewriting the system (1.1) in the abbreviated form $L\phi_n = T_n$, where $T_n = I + k(\phi_{n-1} + \phi_{n+1} - 2\phi_n)$ is interpreted as the torque exerted on the pendulum by its neighbors, we note that T_n is a slowly varying function (which is justified if $k << 1$); this allows us to (cautiously) think of T_n as a constant.

We describe the basic mechanism of the traveling waves. We want ϕ_{N-1} to start and ϕ_N to stop roughly at the same time. It may seem surprising that the starting of ϕ_{N-1} which reduces the backward pull on ϕ_N does not enable ϕ_N to continue running. The reason that this does not happen (for $(I, k) \in D_g$) is the following. Assume that ϕ_{N-1} undergoes a saddle–node bifurcation* when ϕ_N is a finite distance, say, 10π short of its destined sink. Counting from this moment t_{sn}, it takes time $\sim k^{-\frac{1}{2}}$ for ϕ_{N-1} to start (Lemma A.1 from the Appendix), while it takes only $\sim \ln\frac{1}{k}$ for ϕ_N to settle safely in the sink, provided some "bad" parameter values are avoided (Lemma A.2), and thus if k is small, ϕ_{N-1} will start moving only after ϕ_N has settled. This slow starting is due to the *quadratic smallness* of the vectorfield near the saddle–node. We conclude that in order to have one particle run at a time we need a near–simultaneous occurence of the saddle–node for ϕ_{N-1} and of the heteroclinic bifrcation for ϕ_N. It may appear at a first glance that the resulting requirement on I, k is rather stringent, i.e. that (for a fixed k) the value of I would have to be rather specific, but it is not so: for k small, $k^{-\frac{1}{2}} >> \ln\frac{1}{k}$, and this allows us much more freedom in choosing I than might be expected at a first glance. This freedom is explained by the fact that we can allow choices of I for which ϕ_{N-1} will undergo a saddle–node bifurcation long before ϕ_N settles in the sink but start moving only after ϕ_N has settled.

It is also interesting to observe that what one might have expected

* More precisely, the phase plane of an auxiliary system obtained by fixing the two neighbors of ϕ_{N-1} undergoes a saddle–node bifurcation.

to occur in general, namely that the starting of ϕ_{N-1} would prevent ϕ_N from stopping occurs rather for the exceptional parameter values, in the gaps between the domains D_g.

We proceed now to estimate the range of parameters I, k for which the behavior just described could be expected to occur. Assume that at $t = 0$ ϕ_1 is somewhere on its way from J_0 to J_1 with all the left neighbors confined to J_0 and the right neighbors confined to J_1. As ϕ_1 runs, the torque T_1 decreases while T_0 and T_2 grow. If a traveling wave described above is to exist, the following conditions must hold.

(A). *If* ϕ_0 is held fixed artificially in J_0, then ϕ_1 must stop in J_1; this requires that

$$\text{for} \quad \phi_1 \in J_1 \quad \text{we have} \quad T_1 \approx I_h; \tag{3.1}$$

here I_h is the homoclinic value of the torque for the simple pendulum defined below in sec. 3.3.

(B). *If* ϕ_1 reaches J_1 ϕ_0 must start; for that we need to have $T_0 = 1$ before ϕ_1 reaches J_1; we ask that

$$T_0 > 1 \quad \text{for} \quad \phi_1 = 2\pi g - 2\pi. \tag{3.2}$$

(C). ϕ_0 must start *after* ϕ_1 stops – otherwise ϕ_1 will continue running since ϕ_0 will start catching up to ϕ_1 thus decreasing the retarding pull by ϕ_0 on ϕ_1 thus causing the latter to speed up again. As shown in the starting lemma A.1 in the Appendix, ϕ_0 stays near $\frac{\pi}{2}$ for time $> ck^{-\frac{1}{2}}$ after the moment when T_0 reaches the critical saddle–node value $T_0 = 1$; we have thus to make sure that ϕ_1 settles in its sink in J_1 in a shorter amount of time, counting from the moment when $T_0 = 1$. To that end we do not want $T_0 = 1$ to occur when ϕ_1 is still too far from J_1. To be precise, we let $n > 0$ be a large but fixed integer and require that

$$T_0 \leq 1 \quad \text{for} \quad \phi_1 = 2\pi(g - n). \tag{3.3}$$

This estimate guarantees that ϕ_1 will settle in the sink in J_0 in time $\leq ck \ln\frac{1}{k}$, provided the phase point $\Phi_1 = (\phi_1, \dot{\phi}_1)$ does not pass too close to the saddle (it is precisely to avoid this close passage that we remove small neighborhoods in the bifurcation diagram; this results in creation of the wedges in figure 2.1). With k chosen small enough we obtain $t_{\text{starting}} \equiv c_1 k^{-\frac{1}{2}} > \ln\frac{1}{k} \equiv t_{\text{stopping}}$, a condition which assures that ϕ_0 starts only after ϕ_1 has settled into the (slowly moving) sink.

We estimate now the parameters for which the above requirements (3.1)–(3.3) hold.

Using eq. (3.1) and the fact that $\phi_{-1} \in J_0$ and $\phi_1 \in J_1$, we obtain an equivalent estimate

$$I - k2\pi g = I_h + R_1 \quad \text{with} \quad |R_1| < \pi k. \tag{3.1$'$}$$

Next, eq. (3.2) holds if $1 < T_0 = I + k(\phi_{-1} + \phi_1 - 2\phi_0) = I + k(0 + (2\pi g - 2\pi) - 0) + R_2 = I + k2\pi g - k2\pi + R_2$ with $|R_2| < k\pi$; this, in turn, holds if a slightly stronger inequality holds:

$$I + 2\pi k g \geq 1 + \pi k \tag{3.2$'$}.$$

Here again we used $\phi_{-1} \in J_0$ and $\phi_1 \in J_1$. Similarly, we conclude that eq. (3.3) holds provided

$$I + 2\pi g k \leq 1 + 2\pi(n-1)k. \tag{3.3$'$}$$

Combining the requirements (3.1)'–(3.3)' we obtain the estimates on I, g for which (3.1)–(3.3) hold and thus the motions described above should take place:

$$\pi k < I - \frac{1 + I_h}{2} < \pi(n-1)k, \tag{3.4}$$

$$g = \frac{1 - I_h}{4\pi} k^{-1} + O(1). \tag{3.4$'$}$$

This completes our informal discussion.

3.2. An outline of the proof of the main theorem.

Fixing an arbitrary cluster size $N > 1$, we consider the periodic configurations:

$$\phi_{j+N} = \phi_j + 2\pi g, \tag{3.5}$$

where the integer $g = g(\gamma, I, k)$, independent of N, is defined below. With this periodicity condition it suffices to keep track of $\phi = (\phi_1, \dots \phi_N)$ instead of the whole doubly infinite sequence $(\dots, \phi_0, \phi_1, \dots)$; we keep the notation ϕ for the finite sequence as well, without risking a confusion. All the arguments hold for the case $N = \infty$ as well, when only two clusters are present, but we concentrate on the case $N < \infty$. With the periodicity condition (3.5) the phase space of the system $\{(\phi, \dot{\phi})\} = \mathbf{R}^{2N}$. We will construct a Poincaré section Σ of dimension $2N - 1$ in $\mathbf{R}^{2N}$ and show that the flow maps Σ into $\Sigma' = \sigma^{-1}\Sigma = \{(\sigma^{-1}\phi, \sigma^{-1}\psi) : (\phi, \psi) \in \Sigma\}$, where σ is the right shift defined in (2.1) – provided the parameters γ, I and k are chosen appropriately. One can think of Σ' as obtained from Σ by moving each ϕ_j into the position of its right neighbor ϕ_{j+1}. The section Σ itself will be chosen to correspond to the initial conditions with $(\phi_j, \psi_j = \dot{\phi}_j)$, $j = 1, 2, ..., N-1$ lying (roughly speaking) in the potential wells while ϕ_N equals a certain value $\approx \pi g$ half way between the intervals J_0 and $J_1 = J_0 + 2\pi g$. Carrying out this construction would imply the existence of a periodic solution with the properties claimed. Indeed, if the Poincaré map F takes Σ into Σ', then σF takes Σ into itself; hence there exists a fixed point $\Phi_0 \in \Sigma$: $\sigma F \Phi_0 = \Phi_0$; this implies (2.6) in the statement of the theorem. Since $F^N \Phi_0 = \sigma^{-N}\Phi_0$ and $\sigma^{-N}\Phi_0 = \tau^g \Phi_0$ by (3.5), the statement (2.7) of the theorem would follow as well.

A precise idea on the nature of the orbit connecting Σ and Σ' will be seen from the proof.

3.3 Two simple auxiliary systems.

To construct the bifurcation diagram of eq.(1.1) we will need the quantities associated with the well–known equation of a single pendulum with dissipation and a constant torque

$$L\phi \equiv \ddot{\phi} + \gamma\dot{\phi} + \sin\phi = I \tag{3.6}$$

and with a related equation

$$L_K\phi \equiv \ddot{\phi} + \gamma\dot{\phi} + \sin\phi + K\phi = a \tag{3.7}$$

1. There exist two bifurcation values $I = 1$ and $I = I_h = I_h(\gamma)$ for the torque in eq. (3.6), corresponding to the saddle–node bifurcation and to the saddle connection respectively, figure 3.1. For all $I \geq I_h$ eq.(3.6) posesses a unique running periodic solution whose trajectory in the $(\phi, \psi = \dot{\phi})$–plane is given by a periodic function

$$\psi = p(\phi, I) > 0.$$

Further details on this well–understood system can be found in [2], [10], [14].

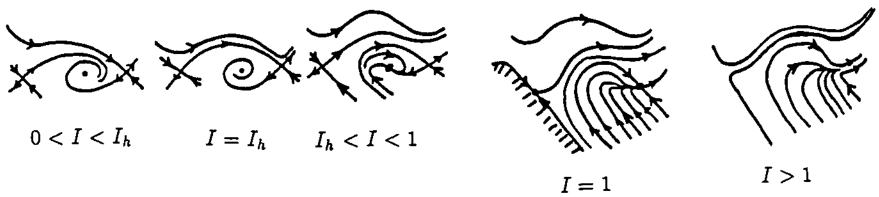

Figure 3.1. The simple pendulum (3.6).

2. Any solution of eq. (3.7) terminates in an equilibrium point; all "moderate" initial conditions terminate in the same equilibrium point, which is a sink, save for the exceptional values of a when the stable manifold of a saddle separating the basins of two sinks passes through the "moderate" set of initial conditions.

To be specific, let $(S(a, K); 0)$ denote the sink in the (ϕ, ψ)– plane which attacts the solution with the initial condition satisfying

$$a - K\phi(0) = 1, \dot{\phi}(0) = 0;$$

we will call the value $S(a; K)$ the distinguished sink. The first saddle equilibrium to the right of $S(a; K)$ will be called the distinguished saddle and denoted by $Sa(a; K)$. When the above initial condition lies on the stable manifold of a saddle, the distinguished sink $S(a; K)$ is undefined; the function S has lines of discontinuity in the (a, K)–plane, given by $a = a_m(K)$, where $m \in \mathbf{Z}$; these are sketched in figure 3.2.

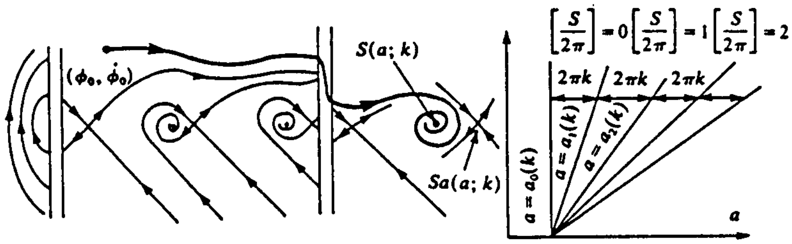

Figure 3.2. Phase portrait of eq.(3.7) and the discontinuity lines of the function $S(a;K)$.

We list the properties of S used in the proof.

1^o Periodicity: $S(a+2\pi K;K) = S(a;K)+2\pi$

2^o $\frac{\partial}{\partial a}S(a;K) = \frac{1}{\cos S+K} \geq c(\gamma) > 0$ where $c(\gamma)$ depends on γ only.

3^o $KS(a;K) = a - I_h + \rho, |\rho| < 3\pi K$.

4^o The discontinuities of $S(a;K)$ form a countable set of curves $a = a_m(K)$, $m \in \mathbf{Z}$ in the (a,K)-plane, figure 3.2. These curves satisfy $a_{m+1}(K) = a_m(K) + 2\pi K$ and $a_m(0) = I_h$.

Proof of $1^o - 4^o$ amounts to the phase plane analysis of eq. (3.7) and is omitted.

3.4. Definition of the section Σ.

In the next paragraph we will define a one–dimensional segment S and two–dimensional boxes B, B' and B'' in the $(\phi,\dot\phi)$–plane. With the aid of these, we define

$$\Sigma = B' \times B \times ... \times B \times B'' \times S, \qquad N \text{ factors in the product.}$$

To define the sets S, B, B' and B'', we fix a constant $r > 0$, to be specified at the end of the proof, and let $B = B_r$ be the r–neighborhood of the sink of the equation $L\phi = I$ in $0 < \phi < \frac{\pi}{2}$, Figure 3.1. The boxes B' and B'' are defined similarly as the r–neighborhoods of the sinks of $L\phi = \frac{1+3I_h}{4}$ and $L\phi = \frac{3+I_h}{4}$ respectively. The centers of the boxes B, B' and B'' thus lie at $\sin^{-1} I$, $\sin^{-1}\frac{1+3I_h}{4}$ and $\sin^{-1}\frac{3+I_h}{4}$ respectively. To define the segment S, we let $\psi =$

$p(I,\phi)$ denote the (unique) periodic solution curve of $L\phi = I$ in the $(\phi, \psi = \dot{\phi})$–plane (figure 3.2), and let $\phi^0 = \pi g$ be half way through the gap (whose length $\approx 2\pi g$ is defined in the next section (3.5)). We let $p_0 = p(\phi^0, \frac{1+I_h}{2})$ and define S as the set $\{(\phi^0,\psi) : \frac{1}{2}p_0 \le \psi \le 2p_0\}$, where I_h is the homoclinic value of I defined in sec. 3.3.

Motivation of the definition of B, B', B''. When ϕ_N is roughly in the middle of its run while the remaining ϕ's are standing, we expect $\phi_1, ..., \phi_{N-1}$ to lie $O(k)$–near the moving sinks* of the systems $L\phi_j = T_j$, $j = 1, ..., N-1$. The boxes B, B' and B'' were chosen as the neighborhoods of these sinks; we proceed to estimate their location when ϕ_N is in the middle of the run, i.e. near πg. For $j = 2, ..., N-2$ the torques $T_j = I + O(k)$, so that ϕ_j should be near $\sin^{-1} I$. For ϕ_{N-1} we have $T_{N-1} = I + \pi g + O(k)$, which via (3.4)' gives $T_{N-1} = \frac{3+I_h}{4} + O(k)$, and $\phi_{N-1} = \sin^{-1}\frac{3+I_h}{4} + O(k)$. Similarly we estimate $\phi_1 = \frac{1+3I_h}{4} + O(k)$.

We conclude with the remark that $\Sigma' \equiv \sigma^{-1}\Sigma = B \times ... \times B \times B'' \times S \times (B' + 2\pi g)$.

3.5. Definition of the gap length g is given by eq. (3.8) below and is motivated as follows. If $\phi_{N\pm1}$ are held artificially fixed in J_0 and J_1 and if ϕ_N is allowed to run then it will tend to** $S(I + k(\phi_{N-1} + \phi_{N+1}); 2k)$ in the non–exceptional cases, i.e. when the first argument of S is not close to $a_m(2k), m \in \mathbf{Z}$. Near the end of the run of ϕ_N we have the estimates $\phi_{N-1} = \frac{\pi}{2} + o(k^0)$ (since ϕ_{N-1} lingers near $\frac{\pi}{2}$ before starting by Lemma A.1 in the Appendix) and $\phi_{N+1} = 2\pi g + \sin^{-1} I + 0(k)$ (since ϕ_{N+1} is safely in the "sink" in J_1 by the time ϕ_N is completing its run). These estimates on its two neighbors suggest that ϕ_N approaches $S(I + k(\frac{\pi}{2} + 2\pi g + \sin^{-1} I); 2k) \equiv S(A; 2k)$, where

$$A = A(I,k,g) = I + k(\frac{\pi}{2} + 2\pi g + \sin^{-1} I);$$

---

* When we speak of the (moving) rest points of a nonautonomous system $\dot{X} = f(X,t)$ we refer to the (t–dependent) zeros of f, which are the rest points of the associated family of autonomous systems $\dot{X} = f(X,\tau)$, τ – *const.*, rather than of the given nonautonomous system.

** The "distinguished sink" $S(a;k)$ as well as its discontinuities $a = a_m(k)$ were defined in sec. 3.3.

on the other hand, we want ϕ_N to enter J_1, i.e. we want $2\pi g < S(A;2k) < 2\pi g + \frac{\pi}{2}$; we choose this, or rather the equivalent requirement

$$G(g,I,k) \equiv [\frac{1}{2\pi} S(A;2k)] = g, \tag{3.8}$$

as the implicit definition of g. The key to understanding the (integer) solutions of this equation is the fact that $[\frac{1}{2\pi} S(A;2k)]$ is constant in A for $a_m(2k) < A < a_{m+1}(2k)$, where $a_{m+1}(2k) - a_m(2k) = 4\pi k$. Plotting the sequence of values of $G(g) = G(g,I,k)$ for different integers g, we observe that as g increases by 1, $G(g)$ increases alternately by 1 and by 0, as explained later in this paragraph. This proves that for any I, k there are exactly two values of g satisfying eq.(3.8). In other words, if eq. (3.8) has one solution g (for fixed I,k), then it must have another, either $g-1$ or $g+1$. Indeed, $S(A,2k)$ has jumps (k is fixed) for A–intervals of length $a_{m+1} - a_m = 4\pi k$, and changing g by one results in changing A by $2\pi k$ which is exacly half of the distance $4\pi k$ between the jumps of $S(\cdot,2k)$. Consequently, if g solves eq. (3.8), we can change it by $+1$ or -1 so as to move A in the direction of the nearest jump thus causing $G(g,I,k)$ to undergo the same change as g and hence preserving the equality (3.8).

To visualize the dependence of the solution g of (3.8) on the parameters I,k (which were fixed in the previous paragraph), we observe that the maximal continuity sets for the solution g of (3.8) are the wedges (cf. Figure 2.1)

$$w'_m = \{(I,k): \quad a_m(2k) < I + k(\frac{\pi}{2} + \sin^{-1} I) < a_{m+1}(2k), \ m \in \mathbf{Z}\},$$

$$w''_m = \{(I,k): \quad a_m(2k) + 2\pi k < I + k(\frac{\pi}{2} + \sin^{-1} I) < a_{m+1}(2k) + 2\pi k, \ m \in \mathbf{Z}\};$$

the overlapping of these wedges reflects the multiple–valued character of $g(I,k)$. We turn $g(I,k)$ into a single–valued function by treating it as defined on the doubly covered half–plane $k > 0$; we can think of one layer as consisiting of the union of the "even" wedges w' and the other layer consisting of the union of the "odd" wedges w''.

In conclusion of this section we note that an approximate value of g is given by $g = \frac{I - I_h}{2\pi k} + O(k^0)$

3.6. Construction of the domains D_g.

Motivated by the heuristic discussion in the beginning of this section, we fix an integer $n > 3$ and consider the triangle

$$\Delta = \{(I,k)\} : \pi k < I - \frac{1+I_h}{2} < \pi(n-1)k, \quad 0 < k < k_0(n) \quad (3.9)$$

where $k_0 = k_0(n)$ is going to be specified later in the proof. We treat Δ as doubly covered; the integer $g(I,k)$ is thus defined uniquely on each sheet; we can think of g as taking even values on top and the odd values on the bottom. Treating each sheet separately, we remove the neighborhoods of the discontinuities of g, as follows. Fixing $\epsilon = \frac{\pi}{100}$, we remove "one percent" of the wedges w'_m, w''_m near their boundaries, obtaining the "shaved" wedges

$$\begin{aligned} W'_m = \{(I,k): \quad & a_m(2k) + \epsilon k < I + k(\frac{\pi}{2} + \sin^{-1} I) \\ & < a_{m+1}(2k) - \epsilon k, \ m \in \mathbf{Z}\}, \end{aligned} \quad (3.10)$$

and similarly W''. We observe that the values of $g(I,k)$ on two adjacent wedges w'_m and w'_{m+1} differ by 2; the same holds for the wedges w''_m and w''_{m+1} on the other sheet, figure 2.1. This shows that all wedges on both sheets can be naturally enumerated by the values of the function $g(I,k)$, as shown in figure 2.1; with this observation, we assign to any integer g that unique wedge W'_m or W''_m on which the function $g(I,k)$ takes that integer value. We will denote that wedge W_g.

Finally, given any $g \in \mathbf{Z}$ we define the domain D_g as the intersection

$$D_g = \Delta \cap W_g.$$

We note that the section of the union $\bigcup_{g=-\infty}^{\infty} D_g$ with the line $k = \ const. < k_0$ consists of at least $2([\frac{(n-2)\pi k}{4\pi k}] - 1) = 2([\frac{(n-2)}{4}] - 1)$ full intervals of length $(4\pi - \epsilon)k$.

In summary, we have to show that with $k_0 = k_0(n)$ chosen sufficiently small, the Poincaré map $F: \ \Sigma \to \Sigma'$ is into for any choice of $(I,k) \in D_g$. In the course of the proof we will also obtain a rather detailed information on the behavior of the solutions.

Remark: widening the range of (I, k).

The statement of the theorem can be strengthened as follows. Taking a sequence $n = 4, 5, \ldots$, we obtain an associated sequence of triangles Δ_n (given by eq.(3.9)) with the widening aperture and with the decreasing heights $k_0(n)$. The union Δ^∞ of all of these triangles contains a wedge shown in figure 2.1 with the right boundary tangent to the I–axis at $I = \frac{1}{2}(1 + I_h)$. The domains D_g can thus be enlarged by replacing Δ with Δ^∞ in the definition. We conclude that for any $n > 3$ there is k so small that the $k = const.$–section of the union of these larger domains consists of at least $2([\frac{(n-2)}{4}] - 1)$ intervals of full length $(4\pi - \epsilon)k$. This shows that many different gap sizes g occur for small k for different values of I.

3.7. Poincaré mapping $F : \Sigma \to \Sigma'$ is into.

Let $\Phi(t)$ be a solution of (1.1) starting at $\Phi(0) \in \Sigma$. Then all ϕ_j, $j = 1, 2, \ldots, N-1$ stand in J_0 and ϕ_N runs for as long as $T_j < 1$ and $T_N > I_h$; the standing of ϕ_j follows from the "parametric nonresonance" lemma A.3 in the Appendix, and the running of ϕ_N follows from the "Stopping Lemma" A.2. In the course of this proof we restrict the range of parameters as dictated by the three lemmas in the Appendix, without mentioning it each time.

Let $t_{\rm sn} > 0$ be the first time when one of the last two inequalities on the torques is violated. Because of our choice of $I > \frac{1}{2}(1 + I_h) + \pi k$, the first inequality to be violated is the one on T_{N-1} (see eq. (3.2), (3.2)' and the discussion), so that we have $T_{N-1}(t_{\rm sn}) = 1$ and hence ϕ_{N-1} is about to start running: we recall that the value $T = 1$ corresponds to the saddle–node bifurcation in the phase plane of the frozen system $L\phi = T$, thus justifying the notation $t_{\rm sn}$. By Lemmas A.1 and A.3 in the Appendix we have $|\phi_{N-1} - \frac{\pi}{2}| < r$ for all

$$t \in [t_{\rm sn}, t_{\rm sn} + c_1 k^{-\frac{1}{2}}], \tag{3.11}$$

with some $c_1 > 0$ independent of k and of I, provided $k < k_0$ for a properly chosen k_0. We claim that ϕ_N is in the r–neighborhood of its sink in J_1 at the end of this interval. Indeed, this follows from the stopping lemma A.2 which is applicable due to our assumptions on the torque I avoiding the bad intervals (eq. (3.10)). It is also clear that the "resting" pendula $\phi_1, \ldots, \phi_{N-2}$ are in the $O(k)$–neighborhoods of their respective sinks at the end of the interval (3.11). We choose now a radius $r > 0$ sufficiently small so that once ϕ_N enters the r–neighborhood of the the sink it is captured, i.e. it approaches an $O(k)$–neighborhood of that sink in time $o(\frac{1}{k})$ (short compared to running time) and stays there for as long as $(0 <)\ T_N < 1 - c$, with some constant $c > 0$. The choice of r can be made independently of k for all $0 < k < k_0$ with some k_0. Such a choice is possible since at the time of capture of ϕ_N we have $T_N = I_h + O(k) < 1 - c'$. It remains now to observe that the condition $T_N < 1 - c$ does, in fact, hold for as long as $(0 <)\phi_{N-1} \leq \phi^0$; we have for such ϕ^0

$$T_N = I - k(2\phi_N - \phi_{N-1} - \phi_{N+1}) < I - k(2(2\pi g) - \phi^0 - 2\pi g) + O(k) =$$

$$I - k(2\pi g - \pi g) + O(k) = I - k(\pi g) + O(k) = \frac{1}{2}(I + I_h) + O(k) < 1 - c;$$

in the last equation we used the estimates (3.1)'–(3.3)". We conclude that at some $t = \tau$ the particle ϕ_{N-1} will reach ϕ^0 while $\Phi_1, ..., \Phi_{N-2}, \Phi_N$ will lie in $B, ..., B'', B' + 2\pi g$ respectively; the latter follows from the discussion in sec. 3.4. This shows that $\Phi(\tau) \in \Sigma'$, q.e.d..

The stability of the resulting traveling wave can be proved in the same way as in [16]. The proof of the existence of the unstable traveling wave is omitted as well as it does not require new ideas beyond those used in the proof above and in the paper just quoted. We note only that the pendula rest at the saddles rather than at the sinks.

The unstable traveling wave behaves as follows: while one pendulum is running, the others rest near the moving saddles rather than the sinks. The proof can be obtained by modifying the constructructions above and using the methods of [16].

Appendix: Three lemmas on stopping and on starting.

The lemmas of this section are the restatements of those proved in [16] (minus some minor misprints). We are looking at a non–autonomous system

$$L_K\phi \equiv \ddot{\phi} + \gamma\dot{\phi} + \sin\phi + K\phi = b(t), \qquad (A.1)$$

together with a "frozen", i.e., autonomous system

$$L_K\phi = a, \quad a = \text{const}.$$

The "starting lemma" states that if $b(t)$ changes slowly: $\dot{b}(t) = O(k)$ and if a solution starts near a sink, then even if a sink disappears in a saddle–node bifurcation due to an increase of b, the solution will stay in a "shadow" of the bifurcated sink for the time $O(k^{-\frac{1}{2}})$.

The "stopping lemma" states that if $b(t)$ stays between two consecutive "bad" values $a_m(K)$, $a_{m+1}(K)$ and not too close to these values, then, given a constant $c > 0$, however large, it takes time $O(\ln\frac{1}{K})$ to travel the distance c from $S(a_m; K) - c$ to a prescribed neighborhood of $S(a_m; K)$.

The statements of lemmas below use a quantity associated with the autonomous equation

$$\ddot{\phi} + \gamma\dot{\phi} + \sin\phi + K\phi = a, \ a = const. \tag{A.2}$$

This quantity is the value of a which gives rise to the saddle-node bifurcation near $\frac{\pi}{2}$; we denote this value by $a_{sn} = a_{sn}(K)$. In other words $a_{sn}(K)$ is defined by the requirement that the equilibrium equation

$$\sin\phi + K\phi = a_{sn}(K)$$

must have a double root near $\frac{\pi}{2}$; one concludes from this that

$$a_{sn}(K) = 1 + \frac{\pi}{2}K + 0(K^2).$$

Lemma A.1 (Starting lemma). *Assume that for some $c > 0$ the right-hand side $b(t)$ satisfies (i), (ii), (iii) below:*

$$|b(0)| \le a_{sn} - c \tag{i}$$

$$|\dot{b}(t)| \le cK \quad \forall t \ge 0 \tag{ii}$$

$$b(t) \le a_{sn} + cK \quad \forall t \ge 0, \tag{iii}$$

and let t_{sn} be the smallest $t > 0$ for which $b(t) = a_{sn}$. Then for any $\gamma > 0$, $\beta > 0$ there exist positive constants $r = r(\gamma, a_{sn} - c)$, $K_0 = K_0(\beta)$, $c_0 = c_0(\beta)$ and c_1 such that for all $0 < K < K_0$, any solution $\phi(t)$ with

$$\Phi(0) \in r-\text{neighborhood of the sink of } L\phi = b(0) \text{ which is nearest } \frac{\pi}{2}, \tag{iv}$$

satisfies

$$|\phi(t) - \frac{\pi}{2}| < \beta \text{ for } t_{sn} \leq t \leq t_{sn} + c_0(\beta)K^{-\frac{1}{2}}, \qquad (A.3)$$

and

$$\phi(t) < \frac{\pi}{2} + c_1 K \text{ for } t \leq t_{sn} \qquad (A.4)$$

The key point of this lemma states that $\phi(t)$ is near $\frac{\pi}{2}$ for the time $\sim \frac{1}{\sqrt{K}}$ after the saddle-node bifurcation. Before stating the next lemma, we recall that in the notation of sec. 3.3, $\psi = p(\phi, I)$ defines the graph in the $(\phi, \dot{\phi})$–plane of the running periodic solution of the forced pendulum equation $\ddot{\phi} + \gamma\dot{\phi} + \sin\phi = I$; p is 2π-periodic in ϕ and is defined only for $I \geq I_h$, where I_h is the homoclinic value of the torque I.

Lemma A.2 (Stopping lemma). *Let* $0 < \epsilon < 1$ *and assume that the right-hand side* b *of eq. (A.1) avoids the bad values:* $a_m(K) + \epsilon\pi K \leq b(t) \leq a_{m+1}(K) - \epsilon\pi K$. *Choose any* $c_1, c_2, > 0$ *and* $c_3 > 2\pi$ *subject to* $I_h + c_1 < 1 - c_1$. *There exist* $r = r(c_1, \gamma)$, $c_4(c_1, c_2)$, $c_5(c_1, c_3)$, $c_6(c_1)$ *and* $K_0 = K_0(c_1, c_2)$ *such that for any solution* $\phi(t)$ *of eq.(A.1) with initial conditions subject to*

$$T_c + c_1 \leq a_m - K\phi(0) \leq 1 - c_1 \qquad (A.5)$$

and

$$|\dot{\phi}(0) - p(\phi(0), a_m - K\phi(0))| \leq r, \qquad (A.6)$$

(i)-(v) below hold for all $0 < K \leq K_0$*:*

(i) $\Phi(t)$ *stays in an* $O(K)$*–neighborhood of the running periodic solution given enough time to settle and while the net torque is larger than* I_h *by a finite amount. More precisely,*

$$|\dot{\phi}(t) - p(\phi(t), a_m - K\phi(t))| \leq c_4 K$$

for all t *when*

$$K\phi(t) \geq K\phi(0) + c_2$$

and

$$b(t) - K\phi(t) \geq I_h + c_2.$$

There exists a (unique) $t' = t'(c_3)$ *such that*

$$\phi(t') = Sa(a_m, K) - c_6; \qquad (ii)$$

There exists $t'' > t'$ *such that*

$$|(\phi(t), \dot{\phi}(t)) - (S(a, K), 0)| < r, \; a = \frac{1}{2}(a_m + a_{m+1}), \text{ for any } t \geq t'', \qquad (iii)$$

$$0 < t'' - t' < c_5 \, ln\frac{1}{\epsilon K}, \qquad t' > \frac{c_6}{K} \qquad (iv)$$

and

$$\phi(t) \leq Sa(\overline{b}; K) + c_1, \quad \forall t \geq 0, \qquad (v)$$

where $\overline{b} = \max_t b(t)$*; the condition on* $b(t)$ *avoiding* $a_m(K)$ *is not necessary for this last statement of the lemma.*

We will also need a simple but crucial "parametric nonresonance" lemma which states roughly that the pendulum (A.1) cannot "start" as long as the net torque on it is < 1 and as long as that torque changes $0(K)$–slowly.

Lemma A.3 (Parametric nonresonance). *If for some* $c > 0$*, the function* $b(t)$ *satisfies the estimates (i), (ii) of Lemma A.1 for all* $0 \leq t \leq T$ *and for all* $K > 0$ *small enough:*

$$b(0) \leq a_{sn} - c$$

$$|\dot{b}(t)| \leq cK \text{ for } 0 \leq t \leq T,$$

together with

$$b(t) \leq a_{sn} - cK \text{ for some } c > 0, \qquad (iii)'$$

then any solution of (A.1) subject to condition (iv) in Lemma A.1 satisfies

$$\phi(t) \leq \frac{\pi}{2} + cK, \; \forall t \in (0, T).$$

Remark. Without the assumption of the slow change of the torque the conclusion of the lemma would be false. We also note that the first two conditions (i), (ii) of Lemma A.1 are the same as in the last lemma, while the third condition $(iii)'$ states, in constrast with (iii) that the saddle–node bifurcation is to be avoided.

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