

Adiabatic Invariants of the Linear Hamiltonian Systems with Periodic Coefficients

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0. HISTORICAL REMARKS

In 1911 Lorentz and Einstein had offered an explanation of the fact that the ratio of energy to the frequency of radiation of an atom remains constant. During the long time intervals separating two quantum jumps an atom is exposed to varying surrounding electromagnetic fields which should supposedly change the ratio. Their explanation was based on the fact that the surrounding field varies extremely slowly with respect to the frequency of oscillations of the atom. The idea can be illustrated by a slightly simpler model—a pendulum (instead of Bohr's model of an atom) with a slowly changing length (slowly with respect to the frequency of oscillations of the pendulum). As it turns out, the ratio of energy of the pendulum to its frequency changes very little if its length varies slowly enough from one constant value to another; the pendulum "remembers" the ratio, and the slower the change the better the memory. I had learned this interesting historical remark from Wasow [15].

Those parameters of a system which remain nearly constant during a slow change of a system were named by the physicists the adiabatic invariants, the word "adiabatic" indicating that one parameter changes slowly with respect to another—e.g., the length of the pendulum with respect to the phase of oscillations.

Precise definition of the concept will be given in Section 1.

In later years the above question was resolved from the point of view of quantum mechanics; however, the mathematical problem remained of interest in itself and it came up in other physical situations such as radiation from antennas and the drift of electromagnetically confined plasmas [15].

Littlewood [8] showed that the above mentioned ratio is an adiabatic invariant—more precisely, for a pendulum

$$\ddot{x} + a^2(\epsilon t)x = 0 \quad (a > 0) \quad (0.1)$$

the change of the ratio of the energy to the frequency is smaller than any

power of ε , if ε is small, provided a (τ) satisfies some additional assumptions, e.g., one can take a (τ) to be a constant outside the interval $[-1, 1]$ and C^∞ .

Wasow [17] strengthened this result. Later Leung and Meyer [7] proved a more general fact—namely, the existence of n independent adiabatic invariants of a slowly varying linear Hamiltonian system in $\mathbb{R}^{2n}\dot{x} = JH(\varepsilon t)x$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I is $n \times n$ identity matrix, $H^T(\tau) = H(\tau)$, whose matrix JH has all eigenvalues purely imaginary and distinct for all τ . Moser [10] observed that there exists at least one adiabatic invariant even if the eigenvalues collide. An important particular case of this observation arises when the Hamiltonian matrix $H(\tau)$ is positive definite.

In this note we consider a more general system—namely, in a slowly varying linear Hamiltonian system we allow in addition a periodic time dependence. We point out that the existence of the adiabatic invariants turns out to be closely related to the theory of strongly stable linear Hamiltonian systems with periodic coefficients.

1. INTRODUCTION; STATEMENT OF THE RESULT; EXAMPLES

We consider a real linear Hamiltonian system

$$\dot{x} = A(t, \varepsilon t, \varepsilon)x, \quad x \in \mathbb{R}^{2n}, \quad (1.1)$$

where A is a Hamiltonian matrix (i.e., $A = JH(t, \varepsilon t, \varepsilon)$ with $H^T = H$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I being $n \times n$ identity matrix), satisfying

- (a) A is periodic in first argument: $A(t + 1, \tau, \varepsilon) = A(t, \tau, \varepsilon)$.
- (b) A is independent of τ for $|\tau| > 1$: $(\partial/\partial\tau)A(t, \tau, \varepsilon) = 0$ for $|\tau| > 1$.
- (c) A is C^∞ for all $(t, \tau, \varepsilon) \in \mathbb{R}^2 \times [0, \varepsilon_0]$ (with some $\varepsilon_0 > 0$).

Equation (1.1) is a model of a periodically excited system which is slowly brought from one state to another; a simple example is a pendulum whose suspension point oscillates periodically in the vertical direction and whose length slowly changes during the time $(-1/\varepsilon, 1/\varepsilon)$ from one constant value to another. For the large times $t < -1/\varepsilon$ or $t > 1/\varepsilon$ (1.1) is a “stationary” system, i.e., it does not depend on “slow” time εt :

$$\dot{x} = A(t, \mp 1, \varepsilon)x = A_\mp x. \quad (1.1)_\mp$$

Assume that with each periodic (stable) system $\dot{x} = A_p x$ we can associate a function $I_{A_p}(x)$ which is an integral of this system. Then $I_{A_\mp}(x)$ are integrals for “past” and “future” systems correspondingly.

Choose a solution $x(t)$ of (1.1), and call $I_{A_-}(x(t))|_{t < -1/\varepsilon} = I_-(\varepsilon)$ and $I_{A_+}(x(t))|_{t > 1/\varepsilon} = I_+(\varepsilon)$.

In general $I_-(\varepsilon) \neq I_+(\varepsilon)$.

DEFINITION. The function $I_{A_p}(x)$ is called an adiabatic invariant of the system $\dot{x} = A(t, \varepsilon t, \varepsilon)x$ iff for any solution $x = x(t, \varepsilon)$ and for any integer $m > 0$

$$I_+(\varepsilon) - I_-(\varepsilon) = I_{A_+}(x(t, \varepsilon))|_{t < -1/\varepsilon} - I_{A_-}(x(t, \varepsilon))|_{t > 1/\varepsilon} = O(\varepsilon^m).$$

Note that the invariant is not defined for the intermediate (nonperiodic) system (for $\varepsilon |t| < 1$).

One might say that an adiabatic invariant is asymptotically an integral. It can be shown, however, that I_- and I_+ do not coincide in general.

We will show in particular, that system (1.1) has at least one nontrivial adiabatic invariant under the assumption that the system with *fixed* slow time

$$\dot{x} = A(t, \tau, \varepsilon)x \quad (\tau \text{ fixed}) \quad (1.1)_\tau$$

is strongly stable for any τ .

The concept of strong stability is described in the next section; we conclude this one by stating the result, giving an example and outlining the method of proof.

THEOREM 1.1. *Let the curve of real periodic Hamiltonian systems (parametrized by τ)*

$$\begin{aligned} \dot{x} &= A(t, \tau, \varepsilon)x, & A &\in C^\infty(\mathbb{R}^2 \times [0, \varepsilon_0]), \\ \partial_\tau A(t, \tau, \varepsilon) &= 0 & \text{for } |\tau| > 1, \end{aligned} \quad (1.1)_\tau$$

lie entirely in the strong stability domain D_p^σ (See Section 2 for definition). Let now the parameter τ change slowly, i.e., consider system (1.1).

Then system (1.1) has at least k independent adiabatic invariants, where k is the number of clusters of symbols in the signature σ , all symbols within each cluster being the same.

Moreover, if in addition a group of eigenvalues corresponding to the cluster of symbols consists of q subgroups disjoint for all τ then there are $(q - 1)$ additional adiabatic invariants.

In particular, if all the eigenvalues are distinct, then there exist n independent adiabatic invariants.

EXAMPLES. 1. As an example we just illustrate the theorem for a system (1.1) with, say, $n = 5$ (i.e., in \mathbb{R}^{10}). Assume that the corresponding systems $(1)_\tau$ all belong to the same stability domain D_p^σ with $\sigma = (+ + - - +)$. We see that the signature consists of three clusters of the same symbol: $(++)$, $(--)$, $(+)$; therefore, we conclude that there are at least three independent adiabatic invariants.

2. As another example, consider a slowly varying Mathieu equation

$$\ddot{x} + (a(\varepsilon t) + b(\varepsilon t) \cos t) x = 0 \quad (1.2)$$

with $a(\tau)$, $b(\tau)$ both C^∞ functions constant for $|\tau| > 1$.

Assume that for all $\tau \in [-1, 1]$ the "frozen" Mathieu equations

$$\ddot{x} + (a(\tau) + b(\tau) \cos t) x = 0 \quad (\tau \text{ fixed}) \quad (1.2)_\tau$$

are strongly stable, i.e., that the curve $(a(\tau), b(\tau))$ belongs to a stability component of Mathieu equation (Fig. 1).

Then (1.2) possesses an adiabatic invariant. Indeed, (1.2) can be written as a system of the form (1.1) with $n = 1$.

The fact that $(a(\tau), b(\tau))$ does not leave a stability component means that the corresponding system does not leave the stability domain. Now, the signature σ of a 1-dimensional system consists of just one symbol: $\sigma = (+)$ or $\sigma = (-)$; applying Theorem 1.1 we obtain the desired statement.

Before concluding this section with a heuristic explanation of the mechanism of adiabatic invariance, we outline the proof of Theorem 1.1.

The main point of the argument is to bring the system into such a normal form that the invariants can be read off immediately, see Wasow [18]. More precisely, we show that system (1.1) is formally equivalent to the Schrödinger equation

$$\dot{z} = D(t, \varepsilon t, \varepsilon) z, \quad D = \sum \varepsilon^k D_k \text{—a formal sum,} \quad D_k^T = -D_k, \quad (1.1)''$$

i.e., that there exists a formal real symplectic transformation $x = Tz$ which preserves real and Hamiltonian character of (1.1),

$$T(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k T_k,$$

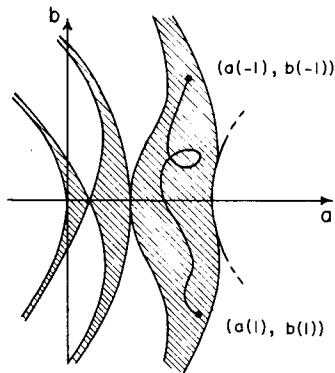


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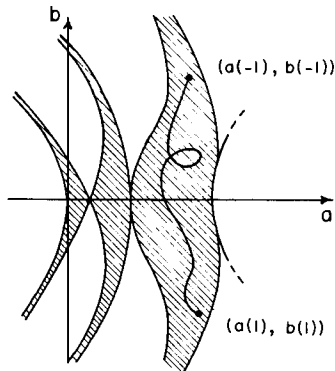


FIGURE 1

which reduces (1.1) to (1.1)". Note that (1.1)" can be written as

$$\dot{z} = i(-iD)z, \quad (-iD)^* = -iD \text{ is Hermitean.}$$

Now, (1.1)" formally has an obvious invariant, namely, the norm, which for (1.1) is an adiabatic invariant. The symplectic character of the transformation will allow us to give a simple geometric interpretation to the adiabatic invariant, see end of Section 6.

We illustrate the mechanism of adiabatic invariance of first order in ε on the example of a slowly changing linear oscillator (0.1).

If $a(\varepsilon t)$ is constant, the phase point (x, \dot{x}) moves along the ellipse $\dot{x}^2/2 + a^2x^2/2 = E$ (E is the total energy of the motion) whose area is $A = E/a = \dot{x}^2/2a + ax^2/2$. This suggests choosing polar coordinates with A as a square of polar radius; namely, we set

$$\begin{aligned} (ax^2/2)^{1/2} &= \sqrt{A} \cos \phi \\ (\dot{x}^2/2a)^{1/2} &= \sqrt{A} \sin \phi, \end{aligned}$$

i.e.,

$$\begin{aligned} x &= (2A/a)^{1/2} \cos \phi \\ \dot{x} &= (2Aa)^{1/2} \sin \phi. \end{aligned} \quad (*)$$

One easily checks that the Jacobian map of this transformation (*) is one, i.e., the map is symplectic. In fact, A, ϕ are just the action-angle variables. Map (*) is, of course, well-defined and symplectic even if a is time-dependent. The oscillator (0.1) expressed in the action-angle variables (*) takes form

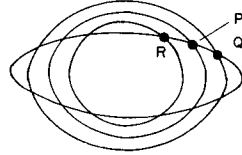
$$\begin{aligned} \dot{A}/A &= \varepsilon(a'/a) \cos 2\phi & (a' = da(\tau)/d\tau) \\ \dot{\phi} &= -a(1 + \varepsilon(a'/2a) \sin 2\phi), \end{aligned}$$

from which the first order adiabatic invariance can easily be read off. Indeed, the second equation states that ϕ changes linearly (up to $O(\varepsilon)$) which implies that in the first equation $\log A$ and thus A changes only by $O(\varepsilon^2)$ during one change of ϕ by 2π , due to the fact that $\cos 2\phi$ oscillates with near zero time-average. Therefore A changes by at most $O(\varepsilon)$ during time $(-1/\varepsilon, 1/\varepsilon)$.

Because of the crucial role of the oscillatory behaviour of A it would be of interest to understand the geometrical reason behind this phenomenon, rather than using an explicit calculation.

Here is a heuristic explanation of the oscillatory character of A . As we have seen above, A is the area of the ellipse $\dot{x}^2/2a + ax^2/2 = A$ with $a = a(\varepsilon, t)$, so that $A = A(P, a)$, where $P = (x, \dot{x})$. Since \dot{P} is tangent to the

ellipse, we have $(d/dt) A(P, a(\epsilon t)) = (\partial A/\partial a) \cdot \epsilon a'$. To be specific, let $a' > 0$; the sign of \dot{A} is determined by $\lim_{\alpha \rightarrow 0} (A(P, a + \alpha) - A(P, a))/\alpha$; therefore, we have to trace the sign of the numerator for the various positions of P on the phase plane. It is easy to see that $A(P, a + \alpha) = A(P, a)$ for $\phi(P) = \pi/4 + o(\alpha^0)$, where ϕ is the angle variable (*) (not the Euclidean angle), see Figure. If the ellipse through P is a circle ($a = 1$) then the statement is obvious; for $a \neq 1$ the ellipse becomes a circle in the action-angle variables; since the change (*) is symplectic, i.e., area-preserving, the statement about the areas is still true. The above discussion and the figure make it clear that



The two ellipses through the point P have equal area, i.e.,

$$A(Q, a + \alpha) > A(P, a) = A(P, a + \alpha) > A(R, a + \alpha).$$

the difference $A(P, a + \alpha) - A(P, a)$ changes sign at approximately $\phi = \pi/4 + (\pi/2)k$, $k = 0, 1, 2, 3$, during one full rotation of P .

We remark in conclusion that the above described mechanism is also responsible for the adiabatic invariance of a system with n degrees of freedom: as it turns out such a system can be decomposed asymptotically into n weakly coupled linear oscillators (under the proper conditions; see [7]).

2. STRONG STABILITY

The more detailed description of the concept can be found in [3–5, 9, 19]; a short exposition is given in [6]. Here we give the results we use, without proofs.

DEFINITION 2.1. The system $\dot{x} = A(t)x$ is called stable, iff for any solution $x(t)$ there exists a constant $C: |x(t)| < C \forall t \in \mathbb{R}$.

DEFINITION 2.2. A linear Hamiltonian system with periodic coefficients

$$\dot{x} = JHx, \quad H^T(t) = H(t) = \bar{H}(t), \quad H(t+1) = H(t) \quad (2.1)$$

is called strongly stable iff for some $\epsilon > 0$ any system $\dot{x} = JKx$, $K^T = K = \bar{K}$, $K(t+1) = K(t)$ with K near $H: \|H - K\| < \epsilon$ is stable. Here $\|\cdot\|$ is any of the norms: C, L^p .

In particular, a strongly stable system is stable.

In other words, the system is strongly stable iff its stability survives small perturbations of the Hamiltonian.

The characteristic property of any real Hamiltonian system is the following: its fundamental matrix $X(t)$, with $X(0) = I$, is symplectic:

$$X^T J X = J,$$

which is equivalent to saying that X preserves an (indefinite) inner product $[x, y] = (Jx, y)$:

$$[Xa, Xb] = [a, b].$$

For that reason symplectic matrices are also called J -orthogonal.

Conversely, if $X(t)$ is a differentiable curve in the space of symplectic matrices, then it is a fundamental solution for some Hamiltonian system.

Symplectic matrices form a group under multiplication, called symplectic group.

For the periodic system (2.1) we consider the matrix $M = X(1)$ (recall, $X(0) = I$), which by the above remark is symplectic.

M is called the monodromy matrix: it provides the linear transformation of a vector in a phase space from its initial position to the position after time 1.

It turns out, that the strong stability of (2.1) is equivalent to the strong stability of its monodromy matrix M .

DEFINITION 2.3. A symplectic matrix M is called strongly stable iff there exists $\varepsilon > 0$ such that any symplectic matrix M_1 near M :

$$|M - M_1| < \varepsilon \quad (|\sim| \text{ is some matrix norm})$$

is stable:

$$|M_j^l| < C \quad \text{for all } j = 0, \pm 1, \pm 2, \dots, \text{ and for some } C \text{ independent of } j.$$

THEOREM 2.1. System (2.1) is strongly stable iff its monodromy matrix is strongly stable.

Next we give the necessary and sufficient conditions for a symplectic matrix to be strongly stable.

Define an indefinite inner product in C^{2n} by

$$[x, y] = \frac{1}{2i} (Jx, y), \quad \text{where } (x, y) = \sum_{j=1}^{2n} x_j \bar{y}_j.$$

The following properties are easily checked:

- (1) $[x, y] = \overline{[y, x]}$
- (2) $[\bar{x}, \bar{y}] = -[y, x]$ —in particular, $[x, x] = 0$ if x is real.

A linear subspace V of C^{2n} is called *positive* (*negative*) iff for any $0 \neq x \in V$ $[x, x] > 0$ (< 0). If V is a positive subspace, then \bar{V} is a negative subspace, according to the property 2) of $[,]$.

A subspace which is either positive or negative we will call *definite*.

We will need a consequence of the

LEMMA 2.1. *The eigenvectors of the stable symplectic matrix corresponding to the different eigenvalues are $[,]$ -orthogonal.*

COROLLARY 2.1. *Two eigenspaces of a stable symplectic matrix corresponding to the different eigenvalues are mutually $[,]$ -orthogonal.*

More generally, if V is a positive invariant subspace, then \bar{V} is a negative invariant subspace, and $[V, \bar{V}] = \{0\}$. The following is the necessary and sufficient condition for the strong stability of a matrix.

THEOREM 2.2. *A symplectic matrix is strongly stable iff each of its eigenspaces is definite.*

COROLLARY 2.2. *A strongly stable matrix does not have real (i.e., +1 or -1) eigenvalues.*

The eigenvalue whose eigenspace is positive (negative) is called an eigenvalue of positive (negative) type.

Remark 2.1. λ is an eigenvalue of positive type iff $\bar{\lambda}$ is an eigenvalue of negative type.

Remark 2.2. Strongly stable matrix can have multiple eigenvalues—provided they are definite—i.e., either positive or negative. Theorem

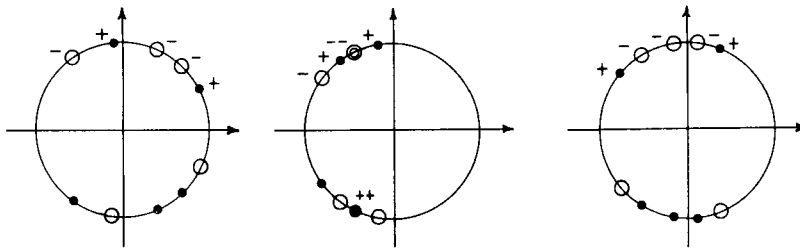


FIGURE 2

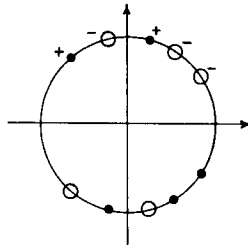


FIGURE 3

2.2 provides the homotopic classification of the set of strongly stable matrices in terms of their spectra.

To any eigenvalue of a strongly stable matrix assign the sign + or - according to whether it is of positive or negative type.

Write the sequence of n symbols + or - corresponding to the eigenvalues on the upper semicircle, counted in counterclockwise direction, any sign written out as many times as the multiplicity of the eigenvalue.

Call this sequence a *signature* of the matrix and denote it by σ . For example, the signature of the matrix with the spectrum on Fig. 3 is $\sigma = (- - + - +)$. Theorem 2.2 implies

THEOREM 2.3. *Two strongly stable matrices can be deformed into each other within the set of strongly stable matrices iff they have the same signature.*

The set of all matrices with the same signature σ is called *stability region*. Theorem 2.3 says that stability regions are connected.

For the future reference we make a simple

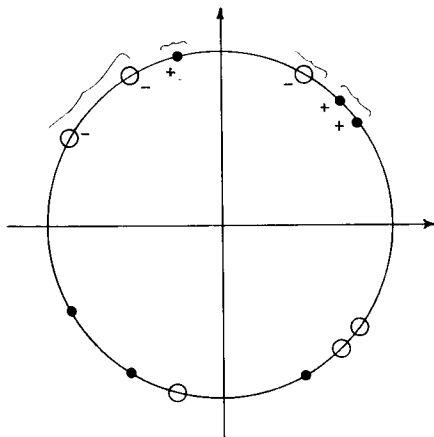


FIGURE 4

Remark 2.3. Let the signature σ consist of, say, k ($1 \leq k \leq n$) clusters of the same sign; e.g., $\sigma = (+ + - + - -)$ consists of four clusters: $++$, $-$, $+$, $--$.

Then the upper half of the spectrum of each matrix in the σ -stability region consists of k groups of eigenvalues, such that no representatives of two different groups coincide (see Fig. 4).

Returning to the strongly stable linear periodic Hamiltonian systems, we pose the question: When are the two such systems deformable into each other without destroying strong stability throughout the deformation?

Clearly, if two systems are homotopic then their monodromy matrices belong to the same stability region. However, this necessary condition is not sufficient.

With every system one can associate an integer p , called index, which together with the signature of the monodromy matrix contains all the information about the homotopic class of the system. This integer can be interpreted as a certain rotation number—roughly speaking it measures how many times the fundamental solution $X(t)$ loops around the “hole” in the group of symplectic matrices during one period, until $X(t)$ reaches the monodromy matrix $X(1)$.

For the definition of the index we refer to either of [4, 6, 19].

THEOREM 2.4. *Two linear periodic Hamiltonian strongly stable systems can be deformed into each other within the class of such systems iff the corresponding indices and signatures of their monodromy matrices are the same.*

DEFINITION 2.4. The set of all systems with the index p and the signature σ is called the stability domain D_p^σ . It is not hard to check that for any integer p and any σ , D_p^σ is nonempty.

3. THE FIRST STEP IN ASYMPTOTIC REDUCTION

According to the outline in the end of Section 1, we transform first system (1.1) to the system of the form

$$\dot{y} = [B(\varepsilon t, \varepsilon) + \varepsilon \tilde{B}_1(t, \varepsilon t, \varepsilon)] y,$$

with the first term containing only slow time-dependence. To construct a transformation we write the fundamental solution $X(t, \tau, \varepsilon)$ of the initial value problem with fixed τ

$$\dot{X} = A(t, \tau, \varepsilon) X, \quad X(0, \tau, \varepsilon) = I, \quad (3.1)$$

in the form given by the Floquet theory:

$$X(t, \tau, \varepsilon) = P(t, \tau, \varepsilon) e^{B(\tau, \varepsilon)t}.$$

It is not hard to see that P, B can be chosen real, smooth functions of their arguments, and P is symplectic.

Indeed, choose $B(\tau, \varepsilon)$ to be a real logarithm of the monodromy matrix of (3.1): $M(\tau, \varepsilon) = e^{B(\tau, \varepsilon)}$. Namely, we take for each τ

$$B(\tau, \varepsilon) = \frac{1}{2\pi} \int_{\Gamma} (M(\tau, \varepsilon) - \lambda I)^{-1} \log \lambda \, d\lambda,$$

Here Γ is the contour (see Fig. 5) containing all the eigenvalues of $M(\tau, \varepsilon)$ (for all τ, ε) in its interior, and not containing the negative real axis. Such a contour exists since the eigenvalues of $M(\tau, \varepsilon)$ depend on τ, ε , $0 \leq \varepsilon \leq \varepsilon_0$, continuously, and are constant in τ for $|\tau| > 1$. Moreover, according to a remark in the previous section, they never cross the real axis since the system (3.1) is strongly stable.

For $\log \lambda$ we take, of course, a real branch of the logarithm:

$$\log \lambda = \log |\lambda| + \theta, \quad |\theta| < \pi.$$

This construction assures the smoothness and reality of $B(\tau, \varepsilon)$, which immediately implies the same properties for $P(t, \tau, \varepsilon)$.

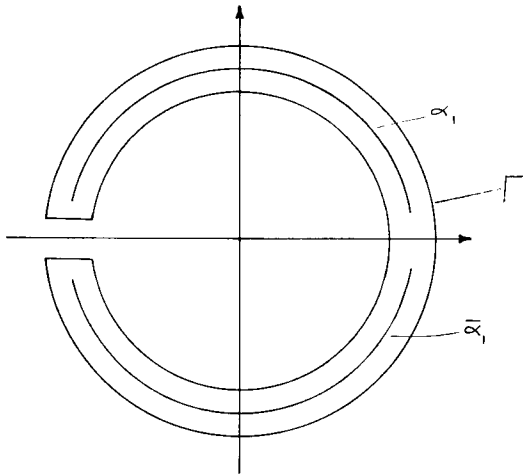


FIG. 5. Spectrum of $M(\tau, \varepsilon)$ is confined to the arcs $\alpha_1, \bar{\alpha}_1$ for all $-\infty < \tau < \infty$ (or equivalently $|\tau| \leq 1$), $0 \leq \varepsilon \leq \varepsilon_0$.

The symplectic character of P is clear from the following observation: $P(1, \tau, \varepsilon) = I$, so that e^B is symplectic, and thus B is Hamiltonian. Therefore, $P(t, \tau, \varepsilon) = X e^{-Bt}$ is symplectic as a product of two symplectic matrices. The transformation

$$x = P(t, \varepsilon t, \varepsilon) y$$

reduces (3.1) to

$$\dot{y} = (P^{-1}AP - P^{-1}P_t - \varepsilon P^{-1}P_\tau) y,$$

i.e.,

$$\dot{y} = [B(\varepsilon t, \varepsilon) + \varepsilon \hat{B}_1] y, \quad (3.2)$$

here $\hat{B}_1 = -P^{-1}P_\tau$.

We remark that \hat{B}_1 is also Hamiltonian, since system (3.2) is Hamiltonian and so is the matrix B .

4. 0th ORDER REDUCTION TO THE SKEW-SYMMETRIC FORM

According to the outline we further transform (3.2) in such a way that the 0th order term becomes skew-symmetric. The transformation has to be real, symplectic.

One would like to compose a transformation matrix out of columns—eigenvectors of $B(\varepsilon t, \varepsilon)$; however, when the eigenvalues of B collide these eigenvectors might become discontinuous. We note, however, that the spectrum of M consists of $2k$ nonintersecting groups (k is the number of clusters of the same symbols in the signature of M). None of these groups collide; moreover, since for $|\tau| > 1$ all eigenvalues are constant in τ , these groups never approach each other closer than some positive distance.

We construct an invariant subspace of $M(\tau, \varepsilon)$ corresponding to each of these groups by setting

$$V_m(\tau, \varepsilon) = \int_{\gamma_m(\tau, \varepsilon)} (M(\tau, \varepsilon) - \lambda I)^{-1} d\lambda \mathbb{C}^{2n}, \quad m = 1, \dots, 2k.$$

where $\gamma_m(\tau, \varepsilon)$ is a contour enclosing each group and not containing other eigenvalues. V_m is obviously also an invariant subspace of B . To show that the dependence of V_m on τ, ε is *smooth*, we can choose the contours $\gamma_m(\tau, \varepsilon)$ piecewise constant in τ, ε . Exactly k of these subspaces will be of positive type; the rest k -complex conjugate of the negative type. Reenumerate V_m so that V_1, \dots, V_k are the positive invariant subspaces of M .

We choose a smooth basis in each $V_m(\tau, \varepsilon)$ and denote it $\varphi_{m1}, \dots, \varphi_{mk_m}$; the

corresponding basis in \bar{V}_m will be the complex conjugate. We obtain thus a basis for C^{2n} .

For the sake of brevity sometimes we omit index m :

$$\varphi_1, \dots, \varphi_n, \bar{\varphi}_1, \dots, \bar{\varphi}_n, \varphi_j = \varphi_j(\tau, \varepsilon).$$

To obtain a symplectic transformation matrix, we transform this basis to a symplectic one, i.e., we find a transformation

$$\psi_{m_j} = \sum_{p=1}^{k_m} a_{jp}^{(m)} \varphi_{mp}, \quad (4.1)$$

leaving each V_m invariant, such that

$$[\psi_j, \psi_l] = (1/2i)(J\psi_j, \bar{\psi}_l) = \delta_{jl} \quad (4.2a)$$

$$[\psi_j, \bar{\psi}_l] = 0. \quad (4.2b)$$

Note that for the matrix Ψ formed out of column vectors $\psi_j, \bar{\psi}_j$

$$\Psi(\tau, \varepsilon) = (\psi_1 \cdots \psi_n \bar{\psi}_1 \cdots \bar{\psi}_n),$$

(4.2) is equivalent to

$$\Psi^T J \Psi = 2iJ. \quad (4.2)'$$

For the sake of brevity we will write (4.1) in the form

$$\psi_j = \sum_{p=1}^n a_{jp} \varphi_p. \quad (4.1)'$$

To find the $n \times n$ matrix A of coefficients of (4.1)' of coefficients of (4.1)' so as to satisfy (4.2) we substitute (4.1)' into (4.2a):

$$\begin{aligned} [\psi_j, \psi_l] &= \left[\sum_{p=1}^n a_{jp} \varphi_p, \sum_{q=1}^n a_{lq} \varphi_q \right] = \sum_p \sum_q a_{jp} \bar{a}_{lq} [\varphi_p, \varphi_q] \\ &= (AQA^*)_{jl}, \end{aligned}$$

where $(Q)_{pq} = [\varphi_p, \varphi_q]$ is $n \times n$ Hermitian matrix: indeed, for any two vectors $x, y \in C^{2n}$ $[x, y] = \overline{[y, x]}$. Equation (4.2a) for A takes form

$$AQA^* = I,$$

or

$$A^*A = Q^{-1}. \quad (4.3)$$

Note, that for any nondegenerate A , the left hand side of (4.3) is positive definite, and the equation has a solution—namely, $Q^{-1/2}$ if and only if Q is positive definite. This condition on B holds precisely due to the fact that we chose φ_j 's as a basis of the positive invariant subspaces.

Indeed, fix any τ, ε and take an $[\cdot, \cdot]$ -orthonormal basis in $V_1 \oplus \cdots \oplus V_k$, denoting it by v_1, \dots, v_n . That this is possible follows from $[V_j, V_l] = \{0\}$ as $j \neq l$ (see Section 2). Thus $\varphi_p = \sum_{j=1}^n T_{pj} v_j$, and

$$[\varphi_p, \varphi_q] = \sum_{j,l} T_{pj} \bar{T}_{ql} [v_j, v_l] = \sum_{j=1}^n T_{pj} \bar{T}_{qj} = (TT^*)_{pq},$$

and positive definiteness of TT^* (since T is clearly nondegenerate) proves the statement.

Finally, we obtain from (4.3), $A(\tau, \varepsilon) = Q^{-1/2}(\tau, \varepsilon)$ (smooth) and (4.2a) is satisfied.

Equation (4.2b) is an automatic consequence of the remark made in Section 2: V_j, \bar{V}_j are $[\cdot, \cdot]$ -orthogonal as two distinct eigenspaces of a stable matrix M .

Finally, the already introduced matrix $\Psi(\tau, \varepsilon)$ is smooth in τ, ε and satisfies (4.2). Ψ is complex; however, ΨR , $R = (1/\sqrt{2}) \begin{pmatrix} I & \\ & -iI \end{pmatrix}$ is real and symplectic.

The transformation

$$y = \Psi(\varepsilon t, \varepsilon) z$$

takes (3.2) into the system

$$\dot{z} = (B_0 + \varepsilon \tilde{B}_1) z, \quad (4.4)$$

where $\tilde{B}_1 = \Psi^{-1} \hat{B}_1 \Psi - \Psi^{-1} \Psi_\tau$ and $B_0 = \Psi^{-1} B \Psi$ is in the block-diagonal form, consisting of $2k$ blocks, since Ψ consists of basis of $2k$ invariant under B subspaces:

$$B_0 = \begin{bmatrix} B_0^1 & & & & & \\ & \ddots & & & & \\ & & B_0^k & & & \\ & & & B_0^1 & & \\ & & & & \ddots & \\ & & & & & B_0^k \end{bmatrix}.$$

The transformation Ψ is complex; so is the resulting system (4.4).

The following lemma implies, in particular, that B_0 is skew symmetric. It establishes the correspondence between the class of real Hamiltonian systems on one hand and their complexifications on the other.

LEMMA 4.1 (Moser [10]). Any real Hamiltonian system $\dot{x} = Ax$ is transformed by

$$Rx = z, \quad R = (1/\sqrt{2}) \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} \quad (4.5)$$

into a complex system

$$\dot{z} = Cz, \quad C = RAR^{-1}$$

with

$$C = \begin{pmatrix} C_1 & C_2 \\ \bar{C}_2 & \bar{C}_1 \end{pmatrix}, \quad \text{where } C_1^* = -C_1, \quad C_2^T = C_2. \quad (4.6)$$

Conversely, (4.6) is a sufficient condition for a complex system to correspond to a real Hamiltonian system via (4.5).

Remark 4.1. The set of all matrices of form (4.6) constitutes a Lie algebra, call it \mathcal{A} . The subset of the block diagonal skew symmetric matrices of the form of B_0 constitutes a subalgebra \mathcal{A}_0 of \mathcal{A} .

We note that the complex system (4.4) is transformed into a real Hamiltonian system by means of transformation $z = R\zeta$. Indeed, the resulting system for ζ will be real Hamiltonian since it can be obtained from (3.1) by $y = \Psi R\zeta$, where by a previous remark ΨR is real symplectic. Thus by the first part of the lemma $B_0^* = -B_0$; our *Oth* goal is achieved.

We will work with complex systems with the matrices subject to (4.6); after the desired formal transformation is found, subsequent application of R will make it real and Hamiltonian, according to Lemma 4.1.

5. FORMAL SKEW-SYMMETRIZATION—HIGHER ORDER

Transform (4.4) further to skew symmetrize higher order (in ε) terms. We make a transformation

$$z = e^{\varepsilon L(t, \varepsilon t, \varepsilon)} w, \quad (5.1)$$

where L is chosen so that the transformed system is still of the form (4.6); as we have already indicated, such a choice of L guarantees the preservation of real and Hamiltonian character of the system.

Also, $L(t, \tau, \varepsilon)$ has to be C^∞ in its arguments and periodic in t .

We need to choose L so that (4.4) transforms by (5.1) into a system with the matrix still in \mathcal{A} ; the following lemma states that it suffices to take $L \in \mathcal{A}$.

LEMMA 5.1. *A system*

$$\dot{x} = Cx \quad (5.2)$$

by a change of variables $x = e^L y$ transforms into

$$\dot{y} = (e^{-L} C e^L - e^{-L} (e^L)_t) y; \quad (5.3)$$

if $A, L \in \mathcal{A}$, then the matrix of (5.3) belongs to \mathcal{A} .

Proof. This lemma is just a complex version of the well-known (and easily checkable) fact that a real symplectic (time-dependent) transformation preserves the real Hamiltonian character of the system; we use this fact to prove the lemma.

Note first that $T = R^{-1} e^L R = e^{R^{-1} L R}$ (R as in Lemma 4.1) is a real symplectic matrix, since $R^{-1} L R$ is real Hamiltonian (Lemma 4.1). Also, $A = R^{-1} C R$ is real Hamiltonian. Hence by the above remark

$$T^{-1} A T - T^{-1} T_t = R^{-1} (e^{-L} C e^{-L} - e^{-L} (e^L)_t) R$$

is a real Hamiltonian matrix, and the application of Lemma 4.1 completes the proof.

Transformation (5.1) takes (4.4) into

$$\dot{w} = (B_0 + \varepsilon \{ [B_0, L] - L_t + \tilde{B}_1 \} + \varepsilon^2 \tilde{B}_2) w, \quad [B_0, L] = B_0 L - L B_0, \quad (5.4)$$

with $\tilde{B}_1, \tilde{B}_2 \in \mathcal{A}$, as follows from the previous lemma. Also an expression in braces belongs to \mathcal{A} ; we have to find $L \in \mathcal{A}$ such that

$$[B_0, L] - L_t + \tilde{B}_1 \in \mathcal{A}_0. \quad (5.5)$$

We solve this equation in the particular case $k = 1$ (i.e., there are two blocks in B_0) and sketch the (similar) proof for $k > 1$.

In notations

$$B_0 = \begin{pmatrix} B^0 & 0 \\ 0 & \bar{B}^0 \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{12} & \tilde{B}_{22} \end{pmatrix}, \quad L = \begin{pmatrix} L^1 & L^0 \\ \bar{L}^0 & \bar{L}^1 \end{pmatrix}$$

(5.5) becomes

$$\begin{pmatrix} B^0 L^1 - L^1 B^0 & B^0 L^0 - L^0 \bar{B}^0 \\ \bar{B}^0 \bar{L}^0 - \bar{L}^0 B^0 & \bar{B}^0 \bar{L}^1 - \bar{L}^1 \bar{B}^0 \end{pmatrix} - L_t + \tilde{B}_1 \in \mathcal{A}_0, \quad (5.5')$$

The necessary and sufficient condition for this to hold is that the upper right block of the left hand side of (5.5)' vanish:

$$B^0 L^0 - L^0 \bar{B}^0 - L_t^0 + B_{12} = 0 \quad (5.6)$$

To obtain a unique solution of (5.5)', we seek $L \in \mathcal{A}$ of the form $L = \begin{pmatrix} 0 & L^0 \\ \bar{L}^0 & 0 \end{pmatrix}$ satisfying smoothness and periodicity conditions. The general solution of (5.6) has the form

$$L^0(t, \tau, \varepsilon) = e^{B^0 t} K_0 e^{-\bar{B}^0 t} - \int_0^t e^{(t-\xi)B^0} B_{12}(\xi, \tau, \varepsilon) e^{-(t-\xi)\bar{B}^0} d\xi.$$

A constant $n \times n$ matrix K_0 has to be chosen so as to satisfy the periodicity requirements

$$L^0(t, \tau, \varepsilon) = L^0(t+1, \tau, \varepsilon);$$

using t -periodicity of B_{12} , we obtain a condition on K_0 , equivalent to the last:

$$e^{B^0} K_0 e^{-\bar{B}^0} - K_0 = \int_{-1}^0 e^{B^0 \eta} B_{12}(\eta, \tau, \varepsilon) e^{-\bar{B}^0 \eta} d\eta,$$

or equivalently,

$$e^{B^0} K_0 - K_0 e^{\bar{B}^0} = C(\tau, \varepsilon), \quad (5.7)$$

where C is a smooth function of its arguments.

Recall that the eigenvalues of B^0 are different from those of \bar{B}^0 —in fact, they are bounded away from each other by some positive distance.

Thus (5.7) has a *unique* solution $K_0(\tau, \varepsilon)$, which is *smooth* in τ, ε , and the same then is true for the corresponding $L(t, \tau, \varepsilon)$.

Finally, $L \in \mathcal{A}$ follows from $(L^0)^T = L^0$, which is an immediate consequence of the invariance of (5.6) under transposition and of the uniqueness of its solution. In the case $k > 1$, e.g., $k = 2$, i.e., when

$$B_0 = \begin{pmatrix} B^1 & 0 & 0 & 0 \\ 0 & B^2 & 0 & 0 \\ 0 & 0 & \bar{B}^1 & 0 \\ 0 & 0 & 0 & \bar{B}^2 \end{pmatrix},$$

we look for

$$L = \begin{pmatrix} L^1 & L^0 \\ \bar{L}^0 & \bar{L}^1 \end{pmatrix} \in \mathcal{A}$$

with

$$L^1 = \begin{pmatrix} 0 & L^2 \\ L^3 & 0 \end{pmatrix} ((L^2)^* = -L^3).$$

Here B^1 and the upper left corner in L_1 are $p \times p$, $0 < p < n$. From (5.5) we obtain the same equation (5.6) for L^0 (B^0 will be just $\begin{pmatrix} B^1 & 0 \\ 0 & B^2 \end{pmatrix}$).

Equation (5.5)' shows that L^2 must satisfy an analogous equation, namely,

$$B^1 L^2 - L^2 B^2 - L_t^2 + B_{11}^2 = 0$$

together with smoothness and periodicity conditions. Here B_{11}^2 is an upper right $p \times (n-p)$ corner in B_{11} .

This equation is solved in the same way as (5.6), and its solvability for periodic smooth L^2 follows from the fact that the eigenvalues of B^1 are bounded away from those of B^2 . The condition $L \in \mathcal{A}$ reduces to $(L^2)^* = -L^3$ and again follows in the same way as for the case $k=1$.

Summarizing, we have solved (5.5) for smooth, periodic: $L \in \mathcal{A}$; according to Lemma 5.1 the matrix of the new system (5.4) still belongs to \mathcal{A} ; the system is of the form

$$\dot{w} = (B_0(\tau, \varepsilon) + \varepsilon B_1(t, \tau, \varepsilon) - \varepsilon^2 \tilde{B}_2) w \quad (5.8)$$

with

$$B_1 = \begin{pmatrix} B_1^1 & & & & & \\ & \ddots & & & & \\ & & B_1^k & & & \\ & & & \bar{B}_1^1 & & \\ & & & & \ddots & \\ & & & & & \bar{B}_1^k \end{pmatrix} \in \mathcal{A}_0$$

skew-symmetric and block diagonal.

The now obvious inductive argument shows the existence for any j (a natural number) of a matrix $L_j = L_j(t, \tau, \varepsilon) \in \mathcal{A}$, smooth and periodic in t , such that system (4.4) is brought to the form

$$\dot{u} = \left(\sum_{m=0}^j \varepsilon^m B_m + \varepsilon^{j+1} \tilde{B}_{j+1} \right) u \quad (5.9)$$

by the transformation $z = e^{\varepsilon L_1} \dots e^{\varepsilon L_j} u$; here

$$B_m \in \mathcal{A}_0 \quad \text{for } m = 0, \dots, j,$$

$$\tilde{B}_{j+1} \in \mathcal{A}.$$

The original system (1.1) is therefore transformed into (5.9) by $x = P\Psi e^{\varepsilon L_1} \dots e^{\varepsilon L_j} u$. There exists therefore a formal transformation

$$T(t, \tau, \varepsilon) = P\Psi e^{\varepsilon L_1} \dots e^{\varepsilon L_j} \dots$$

$$= \sum_{j=0}^{\infty} \varepsilon^j T_j(t, \tau, \varepsilon)$$

bringing system (1.1) to the form

$$\dot{u} = \sum_{j=0}^{\infty} \varepsilon^j B_j u, \tag{5.10}$$

with $B_j \in \mathcal{A}_0$ (i.e., $B_j^* = -B_j$) consisting of $2k$ blocks:

$$B_j = \begin{pmatrix} B_j^1 & & & & & \\ & \dots & & & & \\ & & B_j^k & & & \\ & & & \bar{B}_j^1 & & \\ & & & & \dots & \\ & & & & & \bar{B}_j^k \end{pmatrix}.$$

This proves the formal equivalence of systems (1.1) and (5.10).

Remark 5.1. $TR = \sum \varepsilon^k T^k$ is real symplectic, according to Lemma 4.1; thus TR takes (1.1) into $\dot{u} = Du$,

$$D = \begin{pmatrix} E & -F \\ F & E \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & & 0 \\ & \dots & \\ 0 & & E_k \end{pmatrix}, \quad E_k^T = -E_k, \quad F_k^T = F_k.$$

Remark 5.2. The reduction to a normal form can be carried out in a context more general than that of the Hamiltonian systems. Namely, let \mathcal{A} be a linear subspace of the space Σ of the matrix functions $A(t)$, and let \mathcal{A}_0 be a linear subspace of \mathcal{A} . We wish to state a general condition which would guarantee the reducibility of a system

$$\dot{x} = (B_0 + \varepsilon \tilde{B}_1) x \quad \text{with } B_0 \in \mathcal{A}_0, \quad \tilde{B}_1 \in \mathcal{A} \tag{5.12}$$

to a normal form with the coefficients of the powers of ε belonging to \mathcal{A}_0 .

The first step consists in finding the class of transformations which leave the class of systems with their matrices in \mathcal{A} invariant.

THEOREM 5.1. *Let \mathcal{A}_1 be a linear subspace of Σ . If*

$$D_{\mathcal{A}}\mathcal{A}_1 \subset \mathcal{A}, \quad \text{where } D_{\mathcal{A}}B = [A, B] - B_t, \quad (5.13)$$

then any transformation $x = e^{L_t}y$ with $L \in \mathcal{A}_1$ leaves the class of systems $\dot{x} = C(t)x$ with $C \in \mathcal{A}$ invariant.

Proof. Let $A(s) = e^{-sL}Ce^{sL}$, $B(s) = e^{-sL}(e^{sL})_t$; our aim is to show that $A(1) - B(1) \in \mathcal{A}$ if $C(t) \in \mathcal{A}$ and (5.13) holds. A simple calculation shows that

$$\frac{d^k}{ds^k} A(0) = A_{k+1}, \quad \text{where } A_{k+1} = [A_k, L], \quad A_0 = A, \quad (5.14)$$

and

$$\frac{dB(s)}{ds} = [B, L] - L_t = D_B L \in \mathcal{A} \quad \text{if } B(s) \in \mathcal{A} \text{ with } B(0) = 0. \quad (5.15)$$

Equations (5.14) and (5.15) imply correspondingly: $A(1), B(1) \in \mathcal{A}$. Q.E.D.

Assume now that $\mathcal{A}, \mathcal{A}_1$ are given; the reducibility of (5.12) to the normal form depends on solvability in $L \in \mathcal{A}_1$ of an inclusion $D_{B_0}L + B \in \mathcal{A}_0$ for any $B \in \mathcal{A}$, as we have shown above. This inclusion can be reformulated as follows. Consider a splitting $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}^c$, and let P be a parallel projection on \mathcal{A}^c . Then the last inclusion is solvable iff $PD_{B_0}\mathcal{A}_1 = \mathcal{A}^c$, which together with (5.13) is the desired condition.

In addition to the Hamiltonian systems, this remark can be applied to, say, the reversible ones (see Moser [11] for their description).

6. ADIABATIC INVARIANTS

To prove the existence of adiabatic invariants, we transform (1.1) by the real symplectic transformation

$$x = P\Psi e^{\varepsilon L_1} \dots e^{\varepsilon L_m} Rv = \tilde{T}v;$$

the resulting equation is

$$\dot{v} = \left(\sum_{j=0}^m \varepsilon^j D_j + \varepsilon^{m+1} \tilde{D} \right) v, \quad \tilde{D}(t, \tau, \varepsilon) = 0 \quad \text{for } |\tau| > 1, \quad (6.1)$$

with D_k of the form indicated in the end of the last section:

$$D_j = \begin{pmatrix} E_j & -F_j \\ F_j & E_j \end{pmatrix}, \quad E_j = \begin{pmatrix} E_j^1 & & 0 \\ & \ddots & \\ 0 & & E_j^k \end{pmatrix}, \quad F_j = \begin{pmatrix} F_j^1 & & 0 \\ & \ddots & \\ 0 & & F_j^k \end{pmatrix},$$

$$E_j^T = -E_j, \quad F_j^T = F_j.$$

Consider first a truncation of (6.1):

$$\dot{u} = \sum_{j=0}^m \varepsilon^j D_j u = Du. \quad (6.2)$$

The system (6.2) has k integrals due to its skew and block-diagonal character.

For $|t| > 1/\varepsilon$ (which corresponds to the periodic system, or to no higher order terms in (6.2)) the expressions for these integrals will give us k adiabatic invariants for (1.1).

We first identify these integrals.

Let W_l be an invariant subspace in \mathbb{R}^{2n} of the matrix D , corresponding to the blocks E^l, F^l . Note, that any two such subspaces are (\cdot, \cdot) -orthogonal: $(W_p, W_q) = \{0\}$, $p \neq q$ —it is a trivial consequence of the “block-diagonal” character of D . Let Q_l to be an orthogonal projection on W_l , $l = 1, \dots, k$. Then $|Q_l u|^2$ form a system of k integrals for (6.2). (Their sum is just $|u|^2$ and is an obvious integral.) Using Gronwall’s inequality we easily obtain for the solution of (6.1):

$$v = u + o(\varepsilon^m), \quad \text{where } u \text{ is some solution of (6.2),}$$

or

$$\sup_{\varepsilon|t| \leq 1} ||Q_l v|^2 - |Q_l u|^2| = o(\varepsilon^m);$$

Since $|Q_l u| = \text{const. } \forall t$, and for $\varepsilon|t| > 1$ $|Q_l v| = \text{const.}$ we have

$$|Q_l v|_{t > 1/\varepsilon}^2 - |Q_l v|_{t < -1/\varepsilon}^2 = o(\varepsilon^m), \quad \text{for any } m \geq 0.$$

For the solution $x(t)$ of (1.1) ($x = \tilde{T}v$) this implies

$$|Q_l \tilde{T}^{-1} x|_{t > 1/\varepsilon} - |Q_l \tilde{T}^{-1} x|_{t < -1/\varepsilon} = o(\varepsilon^m),$$

and since for $t > 1/\varepsilon$, $\tilde{T} = P\Psi R = P(t, 1, \varepsilon) \Psi(1, \varepsilon) \Psi = P_+ \Psi_+ R$, and analogously for $t < -1/\varepsilon$, $\tilde{T} = P_- \Psi_- R$ we have

$$|Q_l (P_+ \Psi_+ R)^{-1} x|_{t > 1/\varepsilon}^2 - |Q_l (P_- \Psi_- R)^{-1} x|_{t < -1/\varepsilon}^2 \sim 0$$

as

$$\varepsilon \rightarrow 0, \quad l = 1, \dots, k. \quad (6.3)$$

Thus we obtain

THEOREM 6.1. $I_A(x) = |Q_l P(t, \varepsilon) \Psi(\varepsilon) R x|^2$ is an adiabatic invariant for (1.1), where P, Ψ, Q are built by the periodic matrix $A(t, \varepsilon)$ as we have done earlier.

This proves the existence of k adiabatic invariants and gives their explicit expression (as far as we can find P).

Restricting t to the integers simplifies (6.3), since $P = I$ for integer t :

$$|Q_l R^{-1} \Psi_+^{-1} x|_{t > 1/\varepsilon, t \text{ integer}}^2 - |Q_l R^{-1} \Psi_-^{-1} x|_{t < -1/\varepsilon, t \text{ integer}}^2 \sim 0.$$

These adiabatic invariants have a simple geometric meaning. Consider system (6.1) for $t < -1/\varepsilon$:

$$\dot{v} = D_0(-1, \varepsilon) v. \quad (6.4)$$

In the phase space \mathbb{R}^{2n} of this system there are n invariant 2-dimensional planes $p_j(\varepsilon)$ of the matrix $D_0: D_0 p_j = p_j$, each plane corresponds to a conjugate pair of eigenvalues $\lambda_j, \bar{\lambda}_j$. Moreover, these eigenplanes can be chosen to be $(,)$ -orthogonal to each other—this is an easy consequence of the skew character of D_0 .

These eigenplanes clearly are invariant under the flow of (6.4); the trajectories of (6.4) which remain on these eigenplanes are circles.

Let γ_l be a closed curve in W_l such that its projections on p_j are circles—i.e., trajectories of (6.4), s.t. the orientation of γ induces on its projections orientations coinciding with those given by the flow.

Set $\Gamma_{l,t} = (P(t, -1, \varepsilon) \Psi_-(\varepsilon) R) \gamma$; $\Gamma_{l,t}$ is a closed curve which changes with t periodically—“it breathes.”

The real symplectic character of $P\Psi R$ implies in the notations $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, that

$$\int_{\Gamma_{l,t}} x_2 dx_1 = \int_{\gamma} v_2 dv_1.$$

The right hand side of this relation is the value of the adiabatic invariant we have found, and the left hand side is the sum of oriented areas of projections of $\Gamma_{l,t}$ on 2-dimensional coordinate planes.

7. ASYMPTOTIC EQUIVALENCE

In Section 5 we have shown that the original system (1.1) and the Schrödinger equation (5.10) are formally asymptotically equivalent. Here we use the procedure of Ritt, see Wasow [14] to show the actual asymptotic equivalence of the two systems.

THEOREM 7.1. *Linear Hamiltonian system (1.1) can be transformed by a real symplectic transformation \hat{T} into the Schrödinger equation*

$$\dot{u} = \hat{B}u, \quad \text{with } \hat{B} \sim \sum \varepsilon^k B_k, \quad B_k^T = -B_k, \quad (7.1)$$

with $\hat{T}(t, \tau, \varepsilon)$ 1-periodic in t , smooth in all arguments.

Proof. According to the theorem in previous section a formal transformation $T = \sum \varepsilon^k T^k$ brings (1.1) to (5.10). If there exists a C^∞ function $\hat{T} \sim \sum \varepsilon^k T^k$, then it follows from the construction in the previous section, that in the transformed system $\dot{v} = \hat{B}v$ the matrix-function $\hat{B} \sim \sum \varepsilon^k B_k$. Moreover, $\hat{B} \sim \sum \varepsilon^k B_k$ uniformly in t, τ if $\hat{T} \sim \sum \varepsilon^k T_k$, $\hat{T}_\tau \sim \sum \varepsilon^k T_\tau^k$, $\hat{T}_t \sim \sum \varepsilon^k T_t^k$ uniformly. The proof therefore reduces to showing that there exists a C^∞ function $\hat{T}(t, \tau, \varepsilon)$ on $\mathbb{R}^2 \times [0, \varepsilon_0]$ such that

$$\hat{T} \sim \sum \varepsilon^k T^k, \quad T_\tau \sim \sum \varepsilon^k T_\tau^k, \quad T_t \sim \sum \dots \quad (7.2)$$

uniformly in t, τ .

Set

$$\hat{T}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \alpha_k(\varepsilon) T^k(t, \tau, \varepsilon) \quad (7.3)$$

(see [14]), with

$$\begin{aligned} \alpha_k(\varepsilon) &= 1 && \text{for } k = 0, 1 \\ &= 1 - e^{-1/a_k \varepsilon} && \text{for } k = 2, \dots, \end{aligned}$$

where each a_k is to be chosen so large that the series and its t and τ derivatives converge uniformly. Namely, take

$$a_k > \max_{\substack{(t, \tau, \varepsilon) \in \mathbb{R}^2 \times [0, \varepsilon] \\ 1 \leq j \leq k}} (|T^k|, |\partial_t^j T^k|, |\partial_\tau^j T^k|):$$

then for any $m \geq 0$ and $k \geq \max(2, m)$

$$|\partial_t^m \varepsilon^k \alpha_k(\varepsilon) T^k| < \varepsilon^k \frac{1}{a_k \varepsilon} |\partial_t^m T^k| \leq \varepsilon^{k-1},$$

the same for ∂_τ^m .

This shows that \hat{T} defined by (7.3) is a C^∞ function. Relations (7.2) follows immediately from the uniform convergence of the series (7.3), ∂_t (7.3), ∂_τ (7.3) and the fact that $\alpha_k(\varepsilon) \sim 1$: indeed, for \hat{T} we have:

$$\hat{T} - \sum_{k=0}^N \varepsilon^k T^k = \hat{T} - \sum_{k=0}^N \varepsilon^k \alpha_k(\varepsilon) T^k + \sum_{k=0}^N \varepsilon^k (\alpha_k - 1) T^k.$$

As $\varepsilon \rightarrow 0$, the last term tends to 0 uniformly in t, τ , since $\exists C > 0$: $|T^k(t, \tau, \varepsilon)| < C$ for $(t, \tau, \varepsilon) \in \mathbb{R}^2 \times [0, \varepsilon_0]$. The rest of the relations in (7.2) are established analogously. This concludes the proof of Theorem 7.1.

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