A Simple Explanation of Adiabatic Invariance

MATHEMATICAL CURIOSITIES

By Mark Levi

In the early days of quantum mechanics, there was a sense of wonder that the energy-to-frequency ratio of an atom's radiation is (Planck's) constant, despite external disturbances. Einstein proposed a classical analog of this phenomenon: If we retract the string of an oscillating pendulum (the "atom") slowly and smoothly, then the energy-to-frequency ratio E/ω remains nearly constant (see Figure 1). Einstein's heuristic justification of the near-constancy of E/ω is based on the following nice idea [2]. The tension of the string averaged over a full swing is a bit more than the bob's weight. As we pull the string in, we thus do a bit more work than simply elevating the bob; *this extra work becomes added oscillatory energy.* Translating the previous sentence into the statement $E/\omega \approx \mathrm{const.}$ requires about a page of calculation [2]; a rigorous proof takes considerably longer [1].

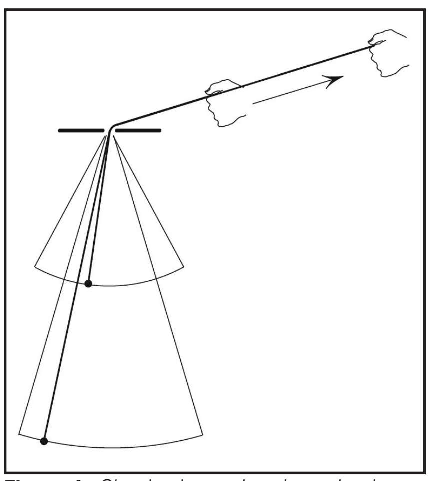

Figure 1. *Slowly shortening the string keeps* $E/\omega \approx \mathrm{const.}$

A justification of adiabatic invariance that is much shorter than Einstein's explanation—also heuristic—recently occurred to me. The main point is the observation that the pendulum's adiabatic invariant E/ω is approximately the angular momentum of another pendulum, of which our given pendulum is a shadow. But the angular momentum of the counterpart pendulum is conserved exactly, which means that E/ω is conserved approximately.

At the root of the derivation of $E/\omega \approx \mathrm{const.}$ is the following trivial observation: *For a circular motion of a point mass $m=1$ with angular velocity ω and kinetic energy E, the angular momentum*

$$AM = 2\frac{E}{\omega}.$$

Indeed, $AM = r(\omega r) = \omega r^2$ and $E = \frac{1}{2}\omega^2 r^2$. If the motion is approximately circular, then

$$AM \approx 2\frac{E}{\omega}. \qquad (1)$$

Justification of $E/\omega \approx \mathrm{const.}$

Along with pendulum 1 in Figure 2, consider a spherical pendulum 2 of the same length that rotates in the horizontal circle of radius r that is equal to the amplitude of pendulum 1. Let us now retract both pendula identically (see Figure 1). For pendulum 2, $AM = \mathrm{const.}$ and AM is the angular momentum around the vertical line through the suspension point; indeed, the torque around that line is zero. Moreover, pendulum 2 continues to move in a near-circular orbit if we retract slowly, so that the key observation in (1) applies:

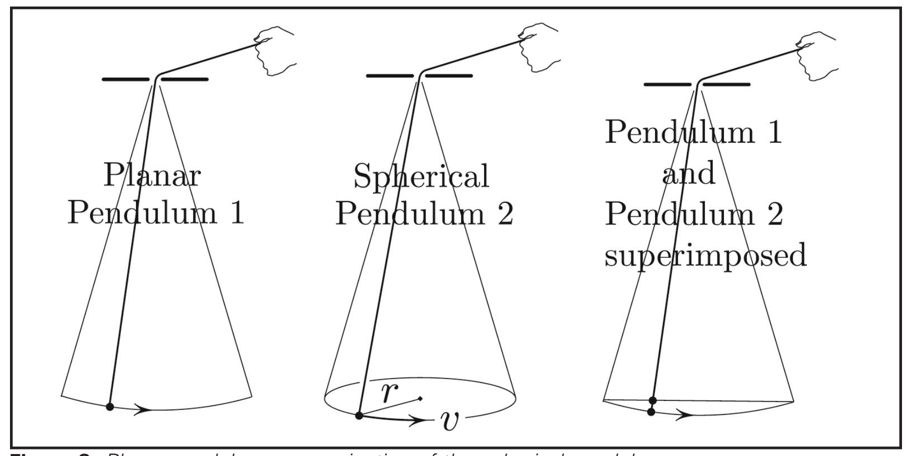

Figure 2. *Planar pendulum as a projection of the spherical pendulum.*

$$AM \approx 2\frac{E_2}{\omega_2}.$$

The subscripts refer to pendulum 2 where E_2 is the kinetic energy. So, we've already found (modulo some rigor) an adiabatic invariant for pendulum 2! But $E_2 \approx E_1$, where E_1 is the total (kinetic + potential[1]) energy of pendulum 1 and $\omega_1 \approx \omega_2$ *if we assume small oscillations*, so that $2\frac{E_1}{\omega_1} \approx AM = \mathrm{const.}$ This concludes our heuristic explanation.

[1] Potential energy is counted as zero at the bottom point of the pendulum's swing.

The figures in this article were provided by the author.

References

[1] Arnold, V.I. (1989). *Mathematical methods of classical mechanics* (2nd ed.). In *Graduate texts in mathematics* (Vol. 60). New York, NY: Springer.

[2] Nakamura, K. (1993). *Quantum Chaos: A new paradigm of nonlinear dynamics*. In *Cambridge nonlinear science series* (Vol. 3). Cambridge, UK: Cambridge University Press.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.