Does Air Rotate with the Tire?

Does the air in a rolling tire rotate at the same speed as the tire, assuming that the tire has been rolling with constant speed for some time? This question is not quite as silly as it seems because the lower part of the tire is flattened by the road, which means that the air in the tire is continuously deformed.

Flow Induced by Squeezing

Imagine squeezing a tire in the vicinity of $\theta = \theta_0$, thus expelling the air particles away from θ_0 and moving them towards the diametrically opposite point $\theta_0 + \pi$. The simplest imaginable expression for the resulting speed is

$$\dot\theta = a\sin(\theta - \theta_0), \qquad (1)$$

where θ stands for the angular coordinate of an air particle. We can think of the tire as a thin tube, like a bike tire. This is certainly not an accurate model, but it gives an excuse for some possibly amusing mathematical observations.

Flow in a Rolling Tire

In Figure 1, a rolling tire is simultaneously squeezed in the forward part of the flattened patch (segment OB) and released in the rear part (segment AO). According to (1), the resulting air motion is given by

$$\dot\theta = a\sin(\theta - \theta_0) - a\sin(\theta - \theta_1) = 2a\sin(\theta - \theta_m).$$

And since the tire is rolling, $\theta_m = \omega t$ (the ground rotates counterclockwise in the tire's frame, as in Figure 1). By taking $2a = 1$ to minimize mess, we then have an angular velocity of air particles *in the wheel's reference frame*:

$$\dot\theta = \sin(\theta - \omega t). \qquad (2)$$

Figure 2 offers some alternative interpretations of (2):

1. θ moves down the gradient of the time-dependent potential $V(\theta, t) = \cos(\theta - \omega t)$. This movement is loosely akin to a cork bobbing on the wave that is traveling to the right with speed ω. Our initial question amounts to estimating the drift of the cork.

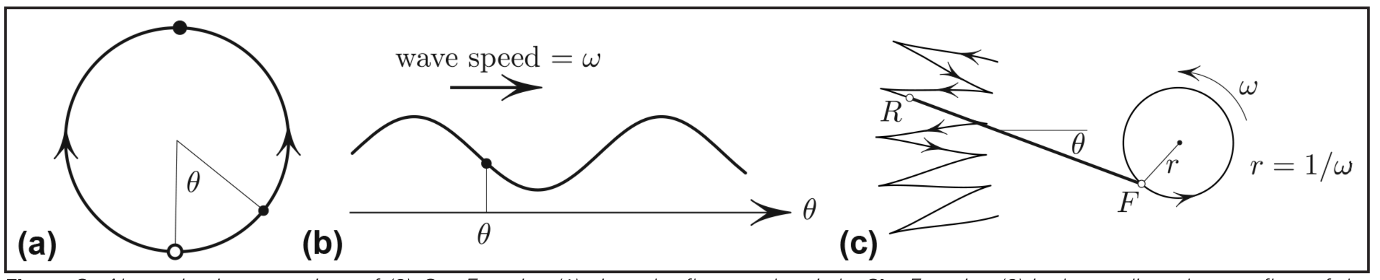

Figure 2. *Alternative interpretations of (2).* **2a.** *Equation (1) gives the flow on the circle.* **2b.** *Equation (2) is the gradient descent flow of the sinusoidal potential $V = \cos(\theta - \omega t)$ that is sliding with speed ω.* **2c.** *θ is the angle between the "bike" RF and a fixed direction as the front F moves in a circle of radius ω^{-1} with angular velocity ω. Each zigzag of R corresponds to one trip of F around the circle.*

2. Consider a "bicycle," i.e., a moving segment RF of fixed length 1 in the plane. The velocity of the "rear" R is constrained to lie along RF so that R cannot "sideslip." Let us guide the "front" F around the circle of radius ω^{-1}, with angular velocity ω and unit speed F. The angle θ that is formed by RF with a fixed direction then satisfies (2).

A naive look at (2) may suggest zero drift, since the average of the right side with respect to both t and θ is zero. However, the "bike" interpretation in Figure 2 is a convincing indication that there *is* drift, i.e., that θ increases with a nonzero average speed.

MATHEMATICAL CURIOSITIES
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Another convincing no-calculation argument for nonzero drift comes from looking at the limiting case of small ω (the opposite of the one in which we are interested). In this case, the wave in Figure 2 moves slowly and a typical solution is trapped by the potential's slowly moving well. The solution thus has the same drift ω as the wave, suggesting that the drift is positive for all ω, including $\omega \gg 1$. For large ω, however, the drift speed is actually a decreasing function of ω, in contrast to the case of small ω.

Estimating the Drift for Large ω

The drift[1] for solutions of (2) for large ω turns out to be

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \frac{1}{2\omega} + O(\omega^{-2}). \qquad (3)$$

So the air in our utterly idealized tire circulates backwards *relative to the tire* with angular velocity $\approx 1/2\omega$. For the ground observer, the air in the rolling tire rotates with the tire but not quite as fast — namely with the angular velocity $\omega - 1/2\omega$. Equivalently, the "cork" on the wave in Figure 2b drifts slowly to the right with speed $1/2\omega$ when ω is large. The faster the wave, the slower the drift.

Proof of (3)

A routine proof of (3)—which I omit so as not to bore the reader—involves putting oneself into a moving frame by setting $\varphi = \theta - \omega t$ as the new dependent variable, thus obtaining an autonomous ordinary differential equation (ODE) that we can then solve in quadratures before expanding the result in powers of ω^{-1}. This method works for any ODE of the form $\dot\theta = v(\theta - \omega t)$ (where v is a periodic function) and shows that the speed of the drift is

$$\frac{\overline{v^2}}{\omega}$$

to the leading order.[2] This expression indeed agrees with (3) for $v = \sin$.

Here, instead, is a short shortcut to (3) that uses the following fact about "bicycles," i.e., unit length segments RF where the velocity of R is constrained to the direction RF: *Let the front F trace a closed path of small diameter δ enclosing area A and returning to the starting point. The segment RF then rotates around F through the angle*

$$\Delta\theta = A + o(\delta^2). \qquad (4)$$

The Prytz planimeter (also called the hatched planimeter)—a simple device that measures areas—is based on this observation. Details about this interesting topic are available in [1].

Now, (4) yields (3) almost immediately. Indeed, one revolution of F around the circle in Figure 2 results in $\Delta\theta = \pi r^2 + o(r^2)$, where $r = 1/\omega$. And the time of one revolution of F is $\Delta t = 2\pi/\omega$, since $\omega \stackrel{def}{=} 2\pi/\Delta t$. The speed of the drift is thus

$$\frac{\Delta\theta}{\Delta t} = \frac{\pi r^2 + o(r^2)}{2\pi/\omega} \stackrel{r=1/\omega}{=} \frac{1}{2\omega} + o(\omega^{-1}),$$

which confirms (3).

I would like to conclude with two puzzles for possible amusement.

Puzzles

Puzzle 1: The speed of the drift is an increasing function ω for small ω and a decreasing function for large ω. Which ω maximizes the drift speed?

On another note, the subject of bikes came up twice already in this article: first in the initial question and second (in a completely different way) in Figure 2. This gives an excuse for another bike tire question that came to mind after I saw a bike at the bottom of the Limmat river in Zurich. Some miscreant probably threw the bike in, and I must reluctantly acknowledge that anonymous vandal's contribution; without him, this puzzle would not have arisen.

Puzzle 2: Assume that the air pressure in the bike tire was initially 2 atmospheres before it was thrown from the bridge and sunk to the depth of 10 meters.[3] Now the tire is squeezed from the outside by an additional pressure of 1 atmosphere. With this added compression, is the new pressure inside the tire $2+1=3$ atmospheres?

The figures in this article were provided by the author.

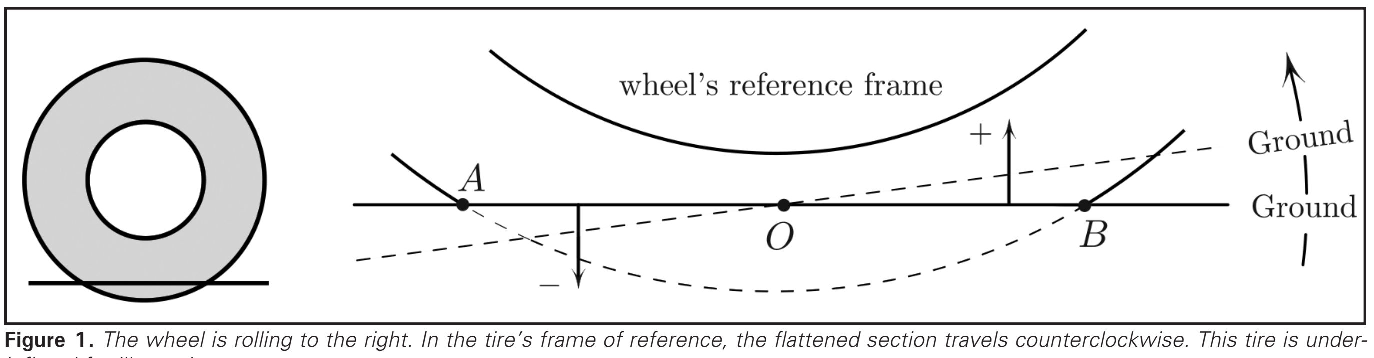

Figure 1. *The wheel is rolling to the right. In the tire's frame of reference, the flattened section travels counterclockwise. This tire is underinflated for illustrative purposes.*

References

[1] Foote, R., Levi, M., & Tabachnikov, S. (2013). Tractrices, bicycle tire tracks, hatchet planimeters, and a 100-year-old conjecture. *Am. Math. Mon., 120*(3), 199-216.

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[1] This is known as Stokes drift.

[2] Here, $\overline{f}$ denotes the average of a function f.

[3] I chose a round number for the depth, but the river was of course less deep.