siam news
Volume 48/ Number 7/ September 2015

A Cycloid is a Tautochrone

A Short Proof

MATHEMATICAL CURIOSITIES
By Mark Levi

In 1659 Christiaan Huygens answered a question that came to him as he watched a swinging chandelier in a church: What curve has the property that a bead sliding along it under uniform gravity and with no friction will oscillate with a period independent of the amplitude?

The answer turned out to be the cycloid generated by a circle, as illustrated in Figure 1. Several solutions to this problem have been found (http://en.wikipedia.org/wiki/Tautochrone_curve); Abel's is particularly remarkable [1].

Presented here is a very short geometrical proof of the tautochronous property of the cycloid. It is based on the fact that $\mathbf{v} \perp \mathbf{r}$, as explained in Figure 1.*

To prove that the cycloid is a tautochrone, it suffices to show that the arclength distance s from the bottom of the cycloid behaves as a harmonic oscillator:

$$a = -ks,$$

where $a = \ddot{s}$, for some constant k. (This idea, which I had learned from Henk Broer, is attributed to Lagrange.) Because $a = 0$ when $s = 0$, we just need to verify that

$$da = -k\, ds. \tag{1}$$

But $da = d(g\cos\theta) = -g\sin\theta\, d\theta$. And from Figure 2 we have $ds = D\sin\theta\, d\theta$. Comparing these expressions for da and ds proves (1) with $k = g/D$. QED

*Incidentally, building on this fact, the line of velocity of every point on a rolling wheel (in the ground reference frame) passes through the topmost point of the wheel. A pebble stuck to the tire always aims straight at, or straight away from, the topmost point of the wheel!

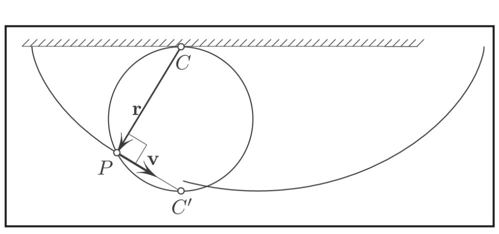

Figure 1. *The contact point C is an instantaneous center of rotation of the rigid wheel, and thus* $\mathbf{v} \perp \mathbf{r}$.

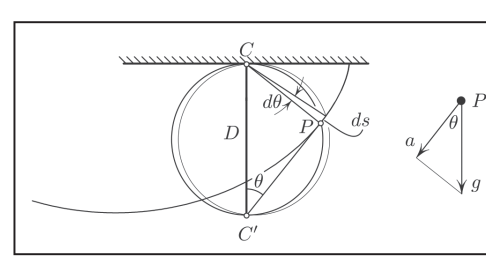

Figure 2. *Proof that* $a = -ks$, *using the fact that* $PC' \perp PC$.

References

[1] M. Levi, *Classical Mechanics with Calculus of Variations and Optimal Control*, AMS, Providence, Rhode Island, 2014.

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ISSN 1557–9573.