Getting Focused

MATHEMATICAL CURIOSITIES
By Mark Levi

A glass cylinder—essentially a thick, flat lens—is neither focusing nor defocusing in the sense that entering parallel beams remain parallel as they pass through, width unchanged.[1]

Let us now cut this slab and open the gaps a bit to obtain a series of lenses (see Figure 1). The alternating focusing and defocusing effects should presumably cancel out, just as they did before we spread the lenses apart. Interestingly, this presumption is wrong; instead, *the gaps turn the neutral slab into a focusing device*. For some mysterious reason, focusing always "wins" over defocusing.

Why Does Focusing Win: a Variational Explanation

It suffices to show that the lens is "optically thicker" between points A and B than between A_1 and B_1, just as a magnifying glass is thicker in the middle than near the edge:

$$T_{A_1B_1} < T_{AB}, \qquad (1)$$

where T is the light's travel time between two points.

To justify (1), it suffices to produce a path between A_1 and B_1 that would take less time than path AB (this path need not be the actual path of light). Figure 1 depicts such a path. Starting with straight line A_1B_1—whose optical length is the same as that of AB—we move the segments passing through the lenses towards the thinner direction of each lens, thus shortening the path's "expensive" part in which the light travels more slowly (we assume that the light travels much slower in the glass). Provided this displacement is not too far, the time savings in the glass will exceed the time gain in the air and the new path will indeed be shorter. In fact, the light's true path will zigzag roughly as shown (with an additional gentle bend away from the axis), according to Snell's law.

[1] In contrast, any optical device that narrows parallel beams is automatically a telescope; it magnifies objects regardless of its internal workings. This is a consequence of the symplectic nature of geometrical optics. More details are available in [1].

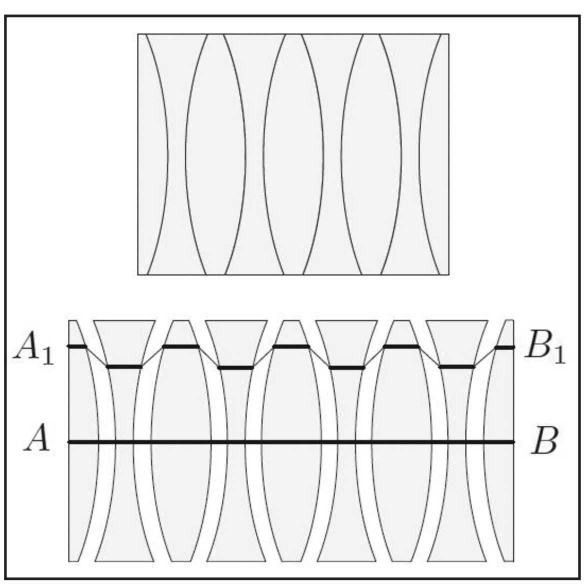

Figure 1. *Separation causes focusing. The optical length of the path A_1B_1 (not a true ray) is shorter than that of AB.*

An Alternative Explanation

Unlike in the previous discussion, we now assume that the lenses are negligibly thin and the break in the ray's slope is linearly proportional to the distance from the optical axis to the point of passage through the lens. In other words, our lenses are Gaussian. When superimposed, the two such lenses cancel out exactly as if they were not there at all. To see why focusing wins upon separation, we convert the problem into a question about matrices.

A ray that enters or exits the lens is described by the pair (x, y), where x is the distance from the point of crossing the lens to the axis and y is the ray's slope (see Figure 2). By the definition of the Gaussian lens, its "transition map"—i.e., assignment of the exit data to the entrance data—is

$$\begin{cases} x_1 = x_0 \\ y_1 = y_0 \pm s x_0 \end{cases} \quad \text{or} \quad z_1 = \begin{pmatrix} 1 & 0 \\ \pm s & 1 \end{pmatrix} z_0 = V_\pm z_0. \qquad (2)$$

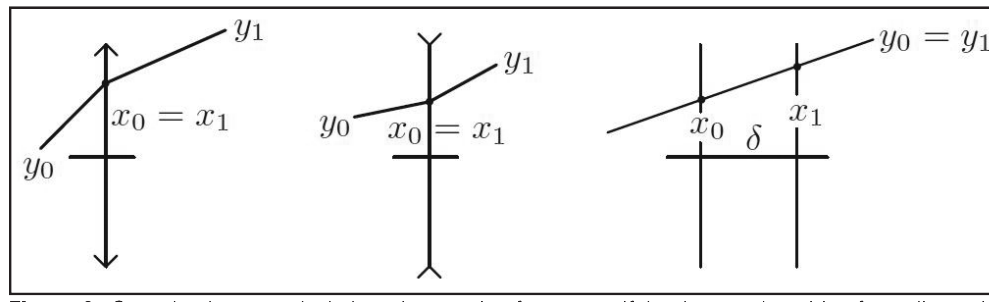

Figure 2. *Gaussian lens: vertical shear is negative for a magnifying lens and positive for a dispersing one. A gap of width δ represents the horizontal shear of strength δ.*

The sign $\pm$ corresponds to defocusing/focusing lenses and the coefficient s has a nice physical meaning: it is the reciprocal of the lens' focal length[2] f, $s = 1/f$. Indeed, an incoming ray parallel to the axis (see Figure 3) refracts and passes through the focus (by definition of the latter). On the other hand, $f = x_1/y_1$ ("run=rise/slope"); but setting $y_0 = 0$ in (2) yields $x_1/y_1 = 1/s$, so that $f = 1/s$ as claimed. The transition map $(x_0, y_0) \mapsto (x_1, y_1)$ is given for the gap of width δ—according to Figure 2—by

$$\begin{cases} x_1 = x_0 + \delta y_0 \\ y_1 = y_0 \end{cases} \quad \text{or} \quad z_1 = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} z_0 = H z_0. \qquad (3)$$

In summary, *a Gaussian lens with focal length f corresponds to the vertical shear of strength $s = 1/f$* with the shear's sign dependent on the lens' status as defocusing or focusing. The empty gap of width δ corresponds with the horizontal shear of strength δ.

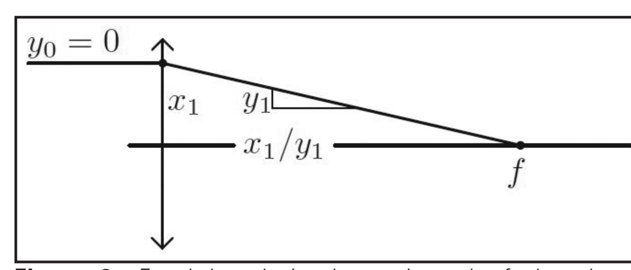

Figure 3. *Focal length is the reciprocal of the shear's strength: $f = 1/s$.*

Let us now consider the sequence "Focusing→Gap→Defocusing→Gap." This combination of lenses corresponds to the product (read from left to right)

$$M = HV_+HV_-,$$

assigning the outgoing data to the incoming data. Multiplication shows that

[2] Normally f is taken to be negative for defocusing lenses, but we do not do this here.

$$\operatorname{tr} M = 2 - (s\delta)^2 < 2. \qquad (4)$$

If we also assume that $\delta < 2f$—the gap is less than twice the focal lengths—then $|\operatorname{tr} M| < 2$. Together with $\det M = 1$, this implies that M is an elliptic rotation; and *this* implies that the combination of lenses F-G-D-G is a focusing device.

It is interesting that we arrived at essentially the same result via two arguments that are so different from one another: the variational one in Figure 1 and the matrix argument. This shows that the purely algebraic result (4) on matrix multiplication is not actually purely algebraic and instead has an alternative variational explanation.

As an aside, a series of lenses gives rise to the product of matrices. A nice undergraduate project would be to build an optical "matrix multiplier" (for 2×2 matrices of determinant one).

The figures in this article were provided by the author.

References

[1] Levi, M. (2014). *Classical Mechanics with Calculus of Variations and Optimal Control: an Intuitive Introduction. Student Mathematical Library* (Vol. 69). Providence, RI: American Mathematical Society.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.