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Oscillatory Escape in a Duffing Equation with a Polynomial Potential

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We show that the time-periodic Hamiltonian systems $d^2x/dt^2 + x^{2n+1} + a(t)\, x^{2l+1} = 0$, $2n > 2l > n$, with a discontinuity in $a(t)$, possess unbounded solutions $x(t)$ which, moreover, oscillate between a finite disk and infinity; in particular $\liminf_{t\to\infty} x(t) < \infty$ and $\limsup_{t\to\infty} x(t) = \infty$. As a consequence, the Poincaré map possesses no invariant KAM curves enclosing the origin outside a bounded disk. © 1997 Academic Press

1. BACKGROUND

Earlier stability results. Stability problem of Hamiltonian dynamics in its most basic but already nontrivial form arises in the study of particles in the force field of a time-periodic potential:

$$\frac{d^2x}{dt^2} + V_x(t, x) = 0, \qquad V(t+1, x) = V(t, x). \tag{1}$$

A large class of such systems turns out to be near-integrable at infinity; as a consequence, "most" large-amplitude solutions are quasiperiodic with two frequencies, and furthermore, all solutions are bounded for all time. The main distinguishing feature of this class of systems is the super-quadratic behavior of the potential V for large $|x|$. The near-integrability at infinity is due to the difference in time scales between the time-dependence of the potential on the one hand and the rapid oscillations of high-amplitude solutions on the other. There has been much work on the

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near-integrablity of (1) in various cases, starting with Morris [MO], who considered the quartic potential $V = x^4 - p(t)\,x$. Later this was extended to arbitrary polynomial potentials of even degree [DZ], [Liu], [LL], [WY] and to more general potentials [L].

It might seem surprising at the first glance that no smallness requirements are made on the time-dependence of $V(x, t)$ which can undergo large changes during one period. For instance, all solutions of both systems

$$\ddot{x} + (2 + \sin t)\, x^3 = 0, \tag{2}$$

$$\ddot{x} + (2 + \sin t) \cosh x = 0 \tag{3}$$

are bounded, and in fact these systems are near-integrable for large-amplitude solutions. This near-integrability result holds for a large class of systems (1) with V satisfying the superquadraticity condition

$$\frac{VV_{xx}}{V_x^2} > \frac{1}{2} + c, \qquad c > 0 \tag{4}$$

together with some extra growth assuptions on higher derivatives [L].

All stability proofs are based on the reduction to a near-integrable form with a subsequent application of Moser's twist theorem [M3].

Smoothness and regularity. For the simplest potential $V = \frac{1}{4}x^4 - p(t)\,x$ corresponding to the system

$$\ddot{x} + x^3 = p(t) \tag{5}$$

considered by Morris [MO], stability holds if $p(t)$ is merely piecewise continuous, or even summable. Dieckerhoff and Zehnder [DZ] proved stability for polynomial potentials with constant leading coefficients; the latter restriction was removed in [LL] with the result that any system (1) with

$$V = \sum_{k=1}^{2n} p_k(t)\, x^k, \qquad n \geqslant 2 \tag{6}$$

is stable as long as all $p_k \in C^{5+\varepsilon}$ and $0 < p_{2n} \in C^{6+\varepsilon}$. There have been further refinements on the degree of smoothness required of different coefficients of polynomials. By modifying proofs in [DZ] and using some approximation techniques, B. Liu ([Liu]) proved that for the system $d^2x/dt^2 + x^{2n+1} + a(t)\,x = p(t)$ the mere continuity of $a(t)$, $p(t)$ implies near-integrability. Recently Wang and You [WY] proved that the smoothness can be lowered to C^2 in a general polynomial potential.

Instability results. If the polynomial nature of the potential is destroyed, the instability can occur even if the potential remains superquadratic in x. The first such "counter-KAM" result was proven by Littlewood [Lit] in 1966 (see also [LO], [L1]), who constructed a system

$$\ddot{x} + V_x(x) = p(t) \tag{7}$$

with $V_x/x \to \infty$ where p is piecewise continuous, with an unbounded solution. Construction of a similar counterexample with a *continuous* p is a much more delicate question; it has recently been solved by Zharnitsky [Z]. In the examples of Littlewood and Zharnitsky the potentials are not polynomial. For the polynomial potentials with their nice x-dependence this leaves open the question of the required smoothness for the coefficients.

2. RESULTS

In this paper we prove that in special polynomial equations

$$\frac{d^2x}{dt^2} + x^{2n+1} + a(t)\, x^{2l+1} = 0, \qquad 2l > n, \tag{8}$$

with $a(t)$ periodic, a jump discontinuity in $a(t)$ causes "chaos": no bounding KAM circles exist near infinity in the Poincaré phase plane. Moreover, we will show the existence of solutions which oscillate "chaotically" between infinity and a bounded disk.

Here, in contrast with the equation (5), the discontinuity destroys the near-integrability. The latter is thus linked with the smoothness of coefficients of higher order terms in the polynomial potential.

In this paper we consider the simplest case of a piecewise constant periodic function $a(t) = k^{[t] \bmod 2}$, but the proof extends to the general case without difficulty. In summary, a jump in the coefficient $a(t)$ destroys all KAM curves bounding the origin from infinity near infinity.

2.1. *Main Theorem*

Before stating the theorem we recall that $a(t) = k^{[t] \bmod 2}$ in the equation (8), so that a alternates periodically between the values 1 and k; we take $0 < k < 1$ throughout. Introduce the Hamiltonian

$$H_k = \tfrac{1}{2} y^2 + x^{2n+2} + kx^{2l+2}. \tag{9}$$

THEOREM 1. *There exist constants λ_0 and $\kappa > 0$ such that for any binary sequence $\sigma = (\sigma_0 \sigma_1 \sigma_2 \cdots)$, $\sigma_j = \pm 1$, satisfying $\sum_{k=0}^{N} \sigma_k \geqslant 0$, $\forall N \in Z^+$, there* *is an initial condition* $z=z_\sigma \in \mathbf{R}^2$ *such that for the Poincaré map* P *of Equation* (8) *we have*

$$H_k(P^i z_\sigma) < H_k(P^{i-1} z_\sigma) - \kappa(H_k(P^{i-1} z_\sigma))^\alpha \qquad \text{if } \sigma_i = -1, \tag{10}$$

$$H_k(P^i z_\sigma) > H_k(P^{i-1} z_\sigma) + \kappa(H_k(P^{i-1} z_\sigma))^\alpha \qquad \text{if } \sigma_i = 1, \tag{11}$$

where $\alpha = (l+1)/(n+1)$.

COROLLARY 2. *By different choices of the sequence we can make a point escape to infinity in infinitely many ways. For instance, we can make the iterates escape to infinity in an oscillatory fashion, so that* $\limsup |P^n z_\sigma| = \infty$, *while* $\liminf |P^n z_\sigma| < \infty$.

COROLLARY 3. *There exists an open interval* I *such that for any* $\mu \in I$ *there exists a sequence* σ *depending on* μ *such that* $|P^n z_\sigma| = O(n^\mu)$.

Remark 4. With a little extra work on can prove a stronger statement: Theorem 1 holds for any periodic $a(t)$ which is piecewise smooth and has at least one jump (but finitely many). For the proof of this statement one has to make a change of variables in which the vectorfield is near-autonomous between the jumps of a. Such changes of variables are readily available in the papers mentioned above.

COROLLARY 5. *Equation* (8) *possesses a solution* $(x(t), \dot{x}(t))$ *with* $|x(t)| + |\dot{x}(t)| \to \infty$ *as* $t \to \infty$, *with* $H_k(x(t))$ *growing as* $t^{1/(1-\rho)}$, *where* $0 < \rho = \rho(k, l, n) < 1$.

Remark 6. It follows from the proof that for the choice $\sigma_i = 1$ for all i we have $|P^n z_\sigma| \approx n^{1/(1-\kappa)}$.

Remark 7. It is somewhat easier to prove that a similar result holds for the equation

$$\ddot{x} + a(t)\, x^{2n+1} + \cdots = 0 \tag{12}$$

with the discontinuity in the highest coefficient (here $\cdots$ denote the lower order polynomial terms). In that case the growth estimates (10) and (11) of Theorem 1 are modified so that they give a geomertic growth rate.

Remark 8. The Poincare map of (8) is the composition of two monotone twist maps; this justifies the application of the Aubry-Mather theory with the result that Poincaré map of Equation (8) possesses a Mather set of any rotation number r in the interval $[r_0, \infty)$ for some r_0. As as a consequence of Theorem 1, none of these Mather sets with $r > r_0$ forms an invariant curve.

2.2. *An Outline of the Proof*

Let P_k be the time one map of the flow with the Hamiltonian H_k given by (9). We note that the Poincaré map is the composition of two twist maps:

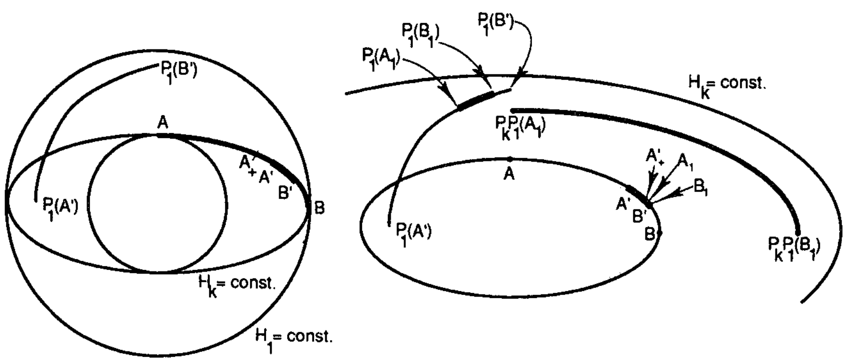
$$P = P_k \circ P_1. \tag{13}$$

The main steps of the proof are illustrated in Fig. 1. We start by choosing a parametrized curve AB given by $z = z(s)$, $s \in [a, b]$, from which all the initial points mentioned in the theorem will be chosen. Referring to Fig. 1, we choose AB in the first quadrant, far enough from the origin, and H_k-monotone, in the sense of the following definition.

DEFINITION 9. We will call a smooth curve $z(s) = (x(s), y(s))$, $s \in [a, b] = I$, with $a \leqslant s \leqslant b$, *H-monotone*, if

$$\frac{d}{ds} H(z(s)) \geqslant 0, \qquad \frac{d}{ds} \arg z(s) < 0. \tag{14}$$

Next we consider the iterated curve $z_1(s) = P_1 z(s)$, Fig. 1. This curve spirals $O(\lambda^{(2l-n)/(2n+2)})$ times around the origin for $a \leqslant s \leqslant b$, where $\lambda = \min_{0 \leqslant s \leqslant \pi/2} H_k(z(s))$. The intersection of this spiral with the *second* quadrant $Q_2 = \{(x, y): x \leqslant 0,\ y \geqslant 0\}$ contains $O(\lambda^{(2l-n)/(2n+2)})$ curves. Each of these curves is H_1-monotone, as we will show later; more precisely, there exist $O(\lambda^{(2l-n)/(2n+2)})$ intervals $I_j \subset [a, b]$ such that each of the curves $z_1(s)$, $s \in I_j$ is H_1-monotone. In a similar way, each of these latter curves maps under P_k onto a spiral which cuts through Q_1 in $O(\lambda^{(2l-n)/(2n+2)})$ H_k-monotone curves. To summarize, we get the existence of subintervals



Iteration of A'B' under P_1 **One more iteration by P_k**

FIG. 1. Iteration of the initial curve AB with the Poincaré map.

$I_{ij} \subset [a, b]$ such that each curve $P_k P_1 z(s)$, $s \in I_{ij}$ is H_k-monotone. This completes an induction step which produces a "chain reaction": one H_k-monotone curve gives rise to many H_k-monotone curves; amongst these curves some are "much closer" to the origin and some are much further than the original curve, as measured by H_k: there exist indices i and j such that

$$H_k(P(z(s))) - H_k(z(s)) \geqslant \kappa H_k(z(s))^\alpha \quad \text{if} \quad s \in I_{ij} \tag{15}$$

and there exist other indices i and j for which

$$H_k(P(z(s))) - H_k(z(s)) \leqslant -\kappa H_k(z(s))^\alpha \quad \text{if} \quad s \in I_{ij}, \tag{16}$$

where $\alpha = (l+1)/(n+1)$ and where $\kappa > 0$ is independent of the initial choice of $z(0)$. We will prove this estimate below. According to this estimate we can control the distance to the origin of the iterate by choosing the initial s in the proper I_{ij}. This allows us to satisfy the first σ_1 in the binary sequence. Applying the above argument to I_{ij} we can satisfy σ_2, and so on. The intersection of the resulting sequence of nested intervals contains the point s_σ whose iterates $P^n z(s_\sigma)$ have the desired itinerary. It remains to mention that in the above argument the distance to the origin has to remain bounded from 0 by a fixed amount for every iterate; this results in the restriction $\sum_1^N \sigma_i \geqslant 0$.

3. PROOFS

3.1. *Notations and Lemmas*

Let P_k be the time one map of the flow with the Hamiltonian (9), and let $T_k(\lambda)$ be the period of the solution of the system with H_k with the energy $H_k = \lambda$.

LEMMA 10. *Let* $H_c = \frac{1}{2}y^2 + V_c(x)$, *with* $V_c(x) = x^{2n+2} + cx^{2l+2}$, *with* $0 < c \leqslant 1$. *If* $c_1 \leqslant c_2$, *we have*

(1) H_{c_1} *monotone curve in* Q_1 *is* H_{c_2} *monotone*;

(2) H_{c_2} *monotone curve in* Q_2 *is* H_{c_1} *monotone.*

It suffices to prove the case (1). Let $z(s) = (x(s), y(s))$, $s \in [a, b]$ be an H^1-monotone curve in Q_1. Note (14) implies that $x_s \equiv (d/ds)\, x(s) > 0$. Indeed, $(d/ds) \arg z(s) < 0$ implies $y_s x < x_s y$, and the contrary assumption $x_s < 0$ implies $y_s < 0$; the last two inequalities contradict the energy estimate $H_s = yy_s + V' x_s > 0$ of (14).

Using this fact we obtain

$$\frac{d}{ds} H_{c_2} = yy_s + V'_{c_2} x_s \geqslant yy_s + V'_{c_1} x_s = \frac{d}{ds} H_{c_1} > 0. \tag{17}$$

The proof of (2) is identical. Q.E.D.

The main theorem of this paper is an easy conclusion of the following main lemma.

LEMMA 11. *Let $z(s)$ be an H_k-monotone curve in Q_1, with $a \leqslant s \leqslant b$. Assume, moreover, that the change in $H_k(z(s)) \equiv h(s)$ is bounded according to*

$$h(s)\big|_{s=a}^{s=b} = h(b) - h(a) \leqslant \tfrac{1}{6} h^{\alpha}(a), \qquad \text{where} \quad \alpha = (l+1)/(n+1). \tag{18}$$

There exist constants $\lambda_0 = \lambda_0(k, n)$ and $\kappa > 0$ such that if $h(a) \geqslant \lambda_0$, then there exist two sub-intervals $I_1 = [a_1, b_1] \subset [a, b]$ and $I^1 = [a^1, b^1] \subset [a, b]$ such that $z_1(s) = P(z(s))|_{s \in I_1}$ and $z^1(s) = P(z(s))|_{s \in I^1}$ are H_k-monotone curves in Q_1. Moreover,

$$\text{for all } s \in I_1 \quad H_k(P(z(s))) \geqslant h(b) + \kappa h^{\alpha}(b) \tag{19}$$

$$\text{for all } s \in I^1 \quad H_k(P(z(s))) \leqslant h(a) - \kappa h^{\alpha}(a). \tag{20}$$

In addition,

$$H_k(P(z(s)))\big|_{s=a_1}^{s=b_1} \leqslant \tfrac{1}{6} h^{\alpha}(a_1). \tag{21}$$

$$H_k(P(z(s)))\big|_{s=a^1}^{s=b^1} \leqslant \tfrac{1}{6} h^{\alpha}(a^1). \tag{22}$$

Inductive application of this lemma according to a given binary sequence σ yields a sequence of nested intervals, $I \supset I_1 \supset I_2 \supset \cdots$ and a sequence of monotone curves $z_i(s) \equiv P^i z(s)$, $s \in I_i$ satisfying

$$\begin{aligned} H_k(P^i z_\sigma) &\leqslant \min_{s \in I_i} \left(H_k(P^{i-1} z(s)) - \kappa (H_k(P^{i-1} z(s)))^{\alpha}\right) \\ &\leqslant H_k(P^{i-1} z_\sigma) - \kappa (H_k(P^{i-1} z_\sigma))^{\alpha} \qquad \text{for negative } \sigma_i, \end{aligned} \tag{23}$$

$$\begin{aligned} H_k(P^i z_\sigma) &\geqslant \max_{s \in I_i} \left(H_k(P^{i-1} z(s)) + \kappa (H_k(P^{i-1} z(s)))^{\alpha}\right) \\ &\geqslant H_k(P^{i-1} z_\sigma) + \kappa (H_k(P^{i-1} z_\sigma))^{\alpha} \qquad \text{for positive } \sigma_i. \end{aligned} \tag{24}$$

We thus conclude that the iterates of any point $z^* \in \bigcap_{n-1}^{\infty} I_n$ escape to infinity[1] in the way described in the Theorem. This completes the proof of the main theorem modulo last lemma.

[1] One can show that this point is in fact unique: $\bigcap_{n=1}^{\infty} I_n = \{x^*\}$.

In order to prove Lemma 11, we first prove:

LEMMA 12. *For Λ_1 sufficiently large, we have, for any $\lambda > \Lambda_1$,*

$$c_1 \lambda^{-n/(2n+2)} < T_k(\lambda) < c_2 \lambda^{-n/(2n+2)}, \; T_k'(\lambda) < -c_3 \lambda^{-1-n/(2n+2)}, \quad (25)$$

where c_1, c_2 and c_3 are positive constants depending only on n, k.

Proof. We omit the simple proof. Q.E.D.

LEMMA 13. *Fix $0 < c < 1$. For any H_c-monotone curve $z(s)$, $s \in [a, b]$ which satisfies*

$$H_c(z(b)) - H_c(z(a)) \leqslant \tfrac{1}{6} H_k(z(a))^{(l+1)/(n+1)}, \quad (26)$$

we have

$$\arg P_c z(s)|_a^b \geqslant \tfrac{1}{2} c_3 H_c(z(a))^{-(n+2)/(2n+2)} (H_c(z(b)) - H_c(z(a))) \quad (27)$$

if $\lambda > \Lambda_1$, where Λ_1, c_3 are the constants in Lemma 12.

Proof. In view of Lemma 12, we have

$$\begin{aligned}
&\arg P_c z(b) - \arg P_c z(a) \\
&= \frac{1}{T_c(H_c(z(b)))} - \frac{1}{T_c(H_c(z(a)))} \\
&\geqslant \frac{T_c(H_c(z(a))) - T_c(H_c(z(b)))}{T_c(H_c(z(a)))^2} \geqslant \frac{T_c(H_c(z(a))) - T_c(H_c(z(b)))}{T_c(H_c(z(a)))^2} \\
&= -\frac{T_c'(H_c(z(a)) + \xi(H_c(z(b)) - H_c(z(a)))}{T_c(H_c(z(a)))^2} (H_c(z(b)) - H_c(z(a))) \\
&\geqslant c_3 \frac{(H_c(z(a)) + \frac{1}{6} H_c(z(a))^{(l+1)/(n+1)})^{-1-n/(2n+2)}}{T_c(H_c(z(a)))^2} (H_c(z(b)) - H_c(z(a))) \\
&\geqslant \frac{1}{2} c_3 H_c(z(a))^{-(n+2)/(2n+2)} (H_c(z(b)) - H_c(z(a))), \quad (28)
\end{aligned}$$

if $H_c(z(a)) > \Lambda_1$. Q.E.D.

Proof of Lemma 11. Let $z(s)$ be an H_k-monotone curve in the first quadrant which satisfies the assumptions in Lemma 11. That is, denoting

$$H_k(z(a)) = \lambda, \quad (29)$$

we assume

$$H_k(z(b)) = \lambda_1 \leqslant \lambda + \tfrac{1}{6}\lambda^\alpha, \tag{30}$$

and $z(a) = (0, y(a))$, $z(b) = (x(b), 0)$ with $\lambda > \lambda_0$; later we will specify λ_0.

In the following, $z(a)$, $z(b)$, $z(a')$, $z(b')$, $z(a_1)$, $z(b_1)$, $z(a_+)$ coincide with A, B, A', B', A_1, B_1, A_+ in the figures respectively.

Following the outline of the proof above, we first prove that the image of $z(s)$ under the map P_1 spirals many times around the origin so that we choose the higher H_k-energy part of it in the second quadrant. In view of Lemma 13, we only need to prove the energy difference between the two end points is sufficiently large.

We first estimate the H_1 energy difference of $z(a)$ and $z(b)$. Note that $H_k(z(a)) \leqslant H_k(z(b))$ by the H_k-monotonicity of $z(s)$, and

$$H_1(z(a)) = H_k(z(a)) \leqslant H_k(z(b)) = x^{2n+2}(b) + kx^{2l+2}(b), \tag{31}$$

$$H_1(z(b)) = x^{2n+2}(b) + x^{2l+2}(b) = H_k(z(b)) + (1-k)\, x(b)^{2l+2}, \tag{32}$$

$$x(b) \geqslant (\tfrac{1}{2})^{1/(2n+2)}\, \lambda^{1/(2n+2)} \tag{33}$$

if $\lambda > 2$. It follows that for $\lambda \geqslant 2$,

$$\begin{aligned} H_1(z(b)) - H_1(z(a)) &\geqslant H_k(z(b)) + (1-k)\, x(b)^{2l+2} - H_1(z(a)) \\ &= H_k(z(b)) - H_k(z(a)) + (1-k)\, x(b)^{2l+2} \\ &\geqslant (1-k)\, x(b)^{2l+2} \geqslant \tfrac{1}{2}(1-k)\, \lambda^\alpha. \end{aligned} \tag{34}$$

In view of Lemma 13, for $\lambda \geqslant \Lambda_1$ we have

$$\begin{aligned} \arg P_1 z(s)|_a^b &\geqslant \tfrac{1}{2}(1-k)\, \lambda^{(l+1)/(n+1)} \cdot \tfrac{1}{2} c_3 \lambda^{-(n+2)/(2n+2)} \\ &= \tfrac{1}{4}(1-k)\, c_3 \lambda^{(2l-n)/(2n+2)}. \end{aligned} \tag{35}$$

If $2l > n$, $\lambda > \Lambda_2 = \max\{\Lambda_1, (96\pi/(1-k)\, c_3)^{(2n+2)/(2l-n)}\}$, then there exists $a_+ \in [a, b]$ such that $\arg P_1 z(s)|_{a_+}^b = 4\pi$ and

$$H_1(z(s))|_{a_+}^b \leqslant \tfrac{1}{12}(1-k)\, \lambda^\alpha. \tag{36}$$

Note that $z(s)$ is H_1-monotone in the first quadrant since it is H_k-monotone, and P_1 is a energy preserving map implies

$$\frac{d}{ds} H_1(P_1 z(s)) = \frac{d}{ds} H_1(z(s)) \geqslant 0.$$

By H_1 monotonicity of $z(s)$, $H_1(z(s'))>H_1(z(s))$ if $s'>s$. By the monotone twist estimate of Lemma 12, which states that the point with large energy rotates clockwise further than the point with small energy, it follows that

$$\arg H_1(z(s'))<\arg H_1(z(s)), \tag{37}$$

if $s'>s$, i.e., $(d/ds)\arg H_1(z(s))\geqslant 0$. By the definition of the H_k-monotonicity, $P_1 z(s)$ is H_k-monotone.

We assure the existence of a subinterval $I'=[a', b']\subset[a_+, b]$ such that $z_1(s)=P_1 z(s)$, $s\in I'$ is H_1-monotone in the second quadrant Q_2:

$$z_1(I')=P_1 z(I')\subset Q_2,\ x_1(a')<0,\ y_1(a')=0;\ x_1(b')=0,\ y_1(b')>0, \tag{38}$$

and

$$\frac{d}{ds}H_1(z_1(s))\geqslant 0, \qquad \frac{d}{ds}\arg z_1(s)<0. \tag{39}$$

From (34) and (36) we obtain, using monotonicity, for $s\in[a', b']$ the following estimate on the energy gain (see (34)):

$$\begin{aligned} H_1(z(s))&\geqslant H_1(z(a_+))=H_1(z(a))+H_1(z(s))|_a^b-H_1(z(s))|_{a^+}^b\\ &\geqslant\lambda+\tfrac{1}{2}(1-k)\,\lambda^\alpha-\tfrac{1}{12}(1-k)\,\lambda^\alpha=\lambda+\tfrac{5}{12}\lambda^\alpha, \end{aligned} \tag{40}$$

if $\lambda>\max\{A_2, 2\}$. To prove that $P_k z_1(s)$, $s\in I'$ spirals many times around the origin, we observe that $z_1(s)$, $s\in I'$ climbs steeply with respect to P_k, see Fig. 1. To be precise,

$$\begin{aligned} H_k(z_1(s))|_{a'}^{b'}&=\frac{y_1^2(b')}{2}-x_1^{2n+2}(a')-kx_1^{2l+2}(a')\\ &=H_1(z_1(b'))-H_1(z_1(a'))+(1-k)\,x_1^{2l+2}(a')\\ &=H_1(z_1(s))|_{a'}^{b'}+(1-k)\,x_1^{2l+2}(a')\geqslant(1-k)\,x_1^{2l+2}(a') \end{aligned} \tag{41}$$

From $H_1(z_1(a'))=x_1^{2n+2}(a')+x_1^{2l+2}(a')\geqslant\lambda$ (see (29)), it follows that $x_1(a')\geqslant(\frac{1}{2})^{1/(2n+2)}\,\lambda^{1/(2n+2)}$ if $\lambda>2$. Thus

$$H_k(z_1(s))|_{a'}^{b'}\geqslant\tfrac{1}{2}(1-k)\,\lambda^\alpha. \tag{42}$$

It follows from Lemma 13 that

$$\begin{aligned}\arg P_k z_1(s)|_{a'}^{b'} &\geqslant \tfrac{1}{4}c_3(1-k)\,\lambda^\alpha H_k(z_1(a'))^{-(n+2)/(2n+2)}\\ &\geqslant \tfrac{1}{8}c_3(1-k)\,\lambda^\alpha H_1(z_1(a'))^{-(n+2)/(2n+2)}\\ &= \tfrac{1}{8}c_3(1-k)\,\lambda^\alpha H_1(z(a'))^{-(n+2)/(2n+2)}\\ &\geqslant \tfrac{1}{8}(1-k)\,c_3\lambda^{(2l-n)/(2n+2)} \qquad (43)\end{aligned}$$

if $\lambda > 2$.

As before, if $\lambda \geqslant 2\Lambda_2$ we can choose the higher-energy subinterval of I' by letting $a'_+ \in (a', b')$ such that $\arg P_k z_1(s)|_{a'_+}^{b'} = 4\pi$ and

$$H_k(z_1(s))|_{a'_+}^{b'} \leqslant \tfrac{1}{12}(1-k)\,\lambda^\alpha. \qquad (44)$$

Thus there exists a subinterval $I_1 = [a_1, b_1] \subset [a'_+, b']$ for which $P_k z_1(s)$ is an H_k-montone curve in the first quadrant; moreover, for $s \in I_1$, using monotonicity, we get

$$\begin{aligned}H_k(Pz(s)) &= H_k(P_k z_1(s)) = H_k(z_1(s))\\ &\geqslant H_k(z_1(a'_+)) = H_k(z(b')) - H_k(z_1(s))|_{a'_+}^{b'}\\ &\geqslant H_1(z_1(b')) - \tfrac{1}{12}(1-k)\,\lambda^\alpha\\ &\geqslant \lambda + \tfrac{5}{12}(1-k)\,\lambda^\alpha - \tfrac{1}{12}\lambda^\alpha = \lambda + \tfrac{1}{3}\lambda^\alpha. \qquad (45)\end{aligned}$$

Together with $\lambda_1 - \lambda \leqslant \frac{1}{6}\lambda^\alpha$, it implies that for $\lambda > 2$

$$H_k(Pz(s)) \geqslant \lambda + \tfrac{1}{3}\lambda^\alpha \geqslant \lambda_1 + \tfrac{1}{6}\lambda^\alpha \geqslant \lambda_1 + \tfrac{1}{7}\lambda_1^\alpha, \qquad (46)$$

as long as $s \in [a_1, b_1]$. Meanwhile, $H_k(Pz(s))|_{a_1}^{b_1} \leqslant H_k(z_1(s))|_{a'_+}^{b'} \leqslant \frac{1}{6}(1-k)\,\lambda^{(l+1)/(n+1)} \leqslant \frac{1}{6}H_k(Pz(a_1))$ if $H_k(z(a)) \geqslant \max\{2, 2\Lambda_2\}$.

Similarly, we can prove the existence of I^1. We omit the virtually identical details. Q.E.D.

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